PREFACE

In the auricular structure introduced by this University for students of Post- Graduate degree programme, the opportunity to pursue Post-Graduate course in Subject introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation.

Keeping this in view, study materials of the Post-Graduate level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Co-operation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing and devising of a proper lay-out of the materials. Practically speaking, their role amounts to an involvement in invisible teaching. For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so mat they may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University.

Needless to add, a great deal of these efforts is still experimental-in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these do admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned.

Professor (Dr.) Subha Sankar Sarkar
Vice-Chancellor

Sixth Reprint : December, 2017

Printed in accordance with the regulations of the Distance Education Bureau of the University Grants Commission.

Subject : Mathematics **Post Graduate**

Paper : PG (MT) : IX A(I)

: Writer :

Prof. P. K. Sengupta

: Editor :

Prof. B. C. Chakraborty

NETAJI SUBHAS OPEN UNIVERSITY

PG (MT)–IX A(I)

Unit 1 ❑ Analytic Continuation

Structure

1.0 Objectives of this Chapter

In this chapter we shall introduce the idea of direct analytic continuation of an analytic function. The concepts of analytic continuation by means of power series, complete analytic function, natural boundary, analytic continuation along a curve will be explained with the help of examples. Homotopic curves, analytic continuation of multi-valued function and Monodromy theorem will also be discussed.

1.1 The idea of analytic continuation

The idea of analytic continuation rests on the notion of analytic function. A function f(z) is analytic at $z = z_0$ if it is differentiable in some $\in$-neighbourhood of z_0 or, equivalently if it can be expressed in the form of a Taylor series in a neighbourhood of that point. The domain of convergence of this power series will be the region of analyticity of the function f(z).

Following Uniqueness Theorem : "If two functions f(z) and g(z), analytic on a region D, are such that f(z) = g(z) on a set $A \subset D$ having a limit point in D, then f(z) = g(z) $\forall z \in D$," we know that if two analytic functions agree in some small neighbourhood of a point situated in their common region of analyticity D, they coincide everywhere in D. We first introduce the idea of analytic continuation by the following examples.

The geometric series

$$1 + z + z^2 + \dots$$

converges for $|z| < 1$ and its sum function $g(z) = \frac{1}{1-z}$ is an analytic function for $|z| < 1$.

The geometric series diverges for $|z| \geq 1$.

However, the function

$$h(z) = \frac{1}{1-z}$$

is analytic for all z except z = 1. But we observe that

$$h(z) = g(z)\ \forall z \in \{|z| < 1 \cap \mathbb{C} \setminus \{1\}\}$$

Thus, we may regard h(z) as determining an analytic continuation of g(z) from the domain $|z| < 1$ into the domain $\mathbb{C} \setminus \{1\}$.

Example 1.1 Consider the Laplace transform of 1 in the z-plane,

$$F(z) = \pounds\ \{1\}(z) = \int_0^\infty e^{-zt}dt = \frac{1}{z} \text{ for } \operatorname{Re}\ z > 0$$

We introduce a function

$$\phi(z) = \frac{1}{z}$$

which is analytic in the complex plance $\mathbb{C}$ except the origin. Here

$$\phi(z) = F(z) \forall z \in \mathbb{C} / (0) \cap \operatorname{Re}\ z > 0$$

and we consider $\phi(z)$ as analytic continuation of F(z) from the domain Re z > 0 into the complex plane with the point z = 0 deleted.

We put these ideas more precisely in the following discussion.

1.2 Direct analytic continuation

Let (i) f(z) and g(z) be analytic functions on domains D_1 and D_2 respectively.

(ii) $D_1 \cap D_2 \neq \phi$

(iii) f(z) = g(z) for all z belonging to $D_1 \cap D_2$

Then g (z) is called a direct analytic continuation of f(z) to D_2, and vice versa.

Theorem 1.1. A direct analytic continuation, if it exists, is unique.

Proof. Let f(z) be an analytic function with domain of definition D_1 and let g(z), another analytic function with domain of definition D_2, be its direct analytic continuation. We shall show that g(z) is unique. On the contrary suppose $\phi(z)$ be another analytic continuation of f(z) into D_2. Then

$$f(z) = g(z) \text{ for all } z \in D_1 \cap D_2$$

Also, $$f(z) = \phi(z) \text{ for all } z \in D_1 \cap D_2$$

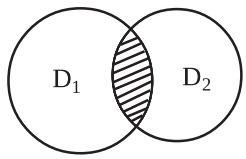


Fig. 1

and so $\phi(z)$ coincides with g(z) in $D_1 \cap D_2$. Thus we have, by the Uniqueness theorem, $\phi(z) = g(z)$ in D_2.

1.3 Analytic continuation of elementary functions

The functions e^z, sin z, cos z, sinh z etc are already known to us. These functions are regular in the entire complex plane. Let us assume, by definition, that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

and observe that it coincides with e^x, known earlier, for real values of z. Thus we can take e^z as the analytic continuation of e^x from real axis into the entire complex plane. Likewise introducing sin z, cos z sinh z, cosh z in the form of power series—

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \quad \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \text{ and } \cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

We can treat them as the analytic continuation of the functions sin x, cos x, sinh x and cosh x respectively from the real axis into the entire complex plane.

1.4 Analytic continuation by power series

We now explain the concept of analytic continuation by means of power series.

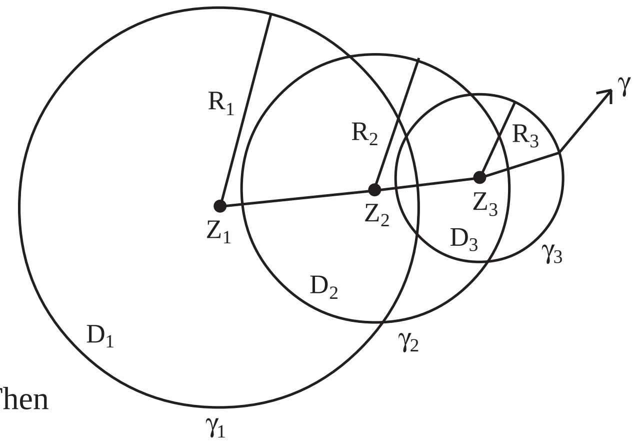
Suppose the initial function $f_1(z)$ is analytic at a point z_1. Then $f_1(z)$ can be represented by its Taylor series about z_1 as

$$f_1(z) = \sum_{n=0}^{\infty} a_n (z - z_1)^n \ldots (1), \text{ where } a_n = \frac{f_1^{(n)}(z_1)}{n!}$$

The circle of convergence γ_1 of the series (1) is given by

$\gamma_1 : |z - z_1| = R_1$, where

$$\frac{1}{R_1} = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}}$$



Let $D_1 = \{z : |z - z_1| < R_1\}$. Then

$f_1(z)$ is analytic in D_1. We draw a curve γ from z_1 and perform analytic continuation along γ as follows :

We take a point z_2 on γ such that the arc $\widehat{z_1 z_2}$ lies inside γ_1.

We then compute the values $f_1(z_2), f_1^1(z_2), \ldots, f_1^{(n)}(z_2)$ by successive term by term differentiation of the series (1) and write

$$f_2(z) = \sum_{n=0}^{\infty} b_n (z - z_2)^n \ldots (2) \text{ where } b_n = \frac{f_1^{(n)}(z_2)}{n!}$$

The circle of convergence γ_2 of the series (2) is given by

$$\gamma_2 : |z - z_2| = R_2, \text{ where } \frac{1}{R_2} = \lim_{n\to\infty} \sup |b_n|^{\frac{1}{n}}$$

Let $D_2 = \{z : |z - z_2| < R_2\}$. Then $f_2(z)$ is analytic in D_2. By uniqueness theorem, $f_1(z) = f_2(z)$ for all $z \in D_1 \cap D_2$. If γ_2 extends beyond γ_1, then $f_2(z)$ gives an analytic continuation of $f_1(z)$ from D_1 to D_2. Similarly, considering a point z_3 on γ such that the arc $\widehat{z_2 z_3}$ lies inside γ_2, we get an analytic function $f_3(z)$ in a circular domain D_3 such that $f_2(z) = f_3(z)$ for all $z \in D_2 \cap D_3$. If D_3 extends beyond D_2, then $f_3(z)$ gives an analytic continuation of $f_2(z)$ from D_2 to D_3. Repeating this process we get a number of different power series representing analytic functions $f_i(z)$ in their respective circular domains D_i which form a chain of analytic continuations of the original function $f_1(z)$ such that (f_i, D_i) is a direct analytic continuation of (f_{i-1}, D_{i-1}).

Note : We may obtain the series (2) from the series (1) in the following way : We rewrite the series (1) in the form : $\sum_{n=0}^{\infty} a_n \{(z - z_2) + (z_2 - z_1)\}^n$

Using binomial theorem we then expand $\{(z - z_2) + (z_2 - z_1)\}^n$ and collect the terms in like powers of $(z - z_2)$ and obtain the series (2).

We give two examples.

Example 1.2 The function

$$f(z) = \frac{1}{1+z^2}$$

possesses two simple poles at $z = \pm i$; Otherwise it is regular throughout the whole complex plane. We first choose a point, say $z = 0$ at which $f(z)$ is analytic and obtain its Taylor series expansion represented by $g(z)$ as

$g(z) = 1 - z^2 + z^4 - \ldots, |z| < 1$

The series fails to converge on and beyond the unit circle, as is clear from the series for $z = 1$ even though the function $f(z)$ is analytic at that point. We can in fact continue the expansion from one region to another. Let us consider a second expansion of $f(z)$, this time about a point $z = \frac{3}{4}$ lying inside the unit circle (i.e. lying inside the region of convergence of the former series). We form the expansion as follows

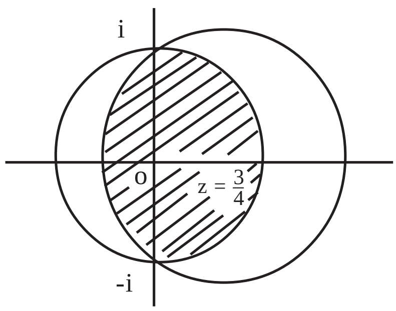


Fig. 2

$$\frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$$

$$= \frac{1}{2i}\left\{\frac{1}{z-\frac{3}{4}+\frac{3}{4}-i} - \frac{1}{z-\frac{3}{4}+\frac{3}{4}+i}\right\}$$

$$= \frac{1}{2i}\left[\frac{1}{\frac{3}{4}-i}\left(1+\frac{z-3/4}{3/4-i}\right)^{-1} - \frac{1}{\frac{3}{4}+i}\left(1+\frac{z-3/4}{3/4+i}\right)^{-1}\right]$$

$$= \frac{1}{2i}[(3/4-i)^{-1}\{1-(z-3/4)/(3/4-i)+(z-3/4)^2/(3/4-i)^2 - \ldots\}$$
$$-(3/4+i)^{-i}\{1-(z-3/4)/(3/4+i)+(z-3/4)^2/(3/4+i)^2 - \ldots\}], \left|z-\frac{3}{4}\right| < \frac{5}{4}$$

$$= \frac{16}{25} - \frac{3}{2}\left(\frac{16}{25}\right)^2\left(z-\frac{3}{4}\right) + \frac{11}{16}\left(\frac{16}{25}\right)^3\left(z-\frac{3}{4}\right)^2 + \frac{21}{16}\left(\frac{16}{25}\right)^4\left(z-\frac{3}{4}\right)^4 \quad \ldots (2)$$

We denote this expansion by h(z), which converges in the right-hand circle $\left|z-\frac{3}{4}\right| < \frac{5}{4}$ and coincides with g(z) in the shaded region. We see that h(z) is clearly a direct analytic continuation of g(z).

Let us construct another analytic continuation of g(z). Now we consider a neighbourhood of the point z = 1 (though it is a boundary point of the unit circle the function f(z) is analytic there) and obtain an expansion represented by

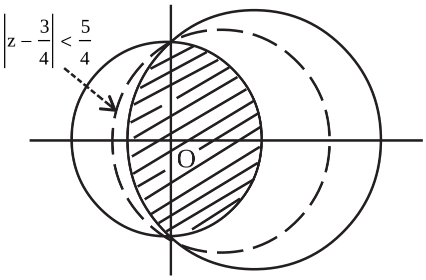


Fig. 3

$$\phi(z) = \frac{1}{2} - \frac{1}{2}(z-1) + \frac{1}{4}(z-1)^2 - \ldots$$

$$\text{for } |z-1| < \sqrt{2} \ldots (3)$$

In this way we can determine all possible direct analytic continuations of g(z) and then continuations of these continuations and so on. A **complete analytic function** is defined as consisting of the original function and the collection of all the continuations so achieved.

Here the complete analytic function is $\frac{1}{1+z^2}$, defined in the whole complex plane barring the points z = ±i.

Example 1.3 Consider the function

$$f(z) = \frac{1}{1+z}$$

Clearly this function is analytic everywhere except at z = – 1. We take a function

$$\phi(z) = 1 - z + z^2 \qquad \text{... (4)}$$

Then sum function $\phi(z)$ is $\dfrac{1}{1+z}$ in | z | < 1. Take a point z = – 1/4 inside the region of convergence of $\phi(z)$ and in a neighbourhood of this point we determine

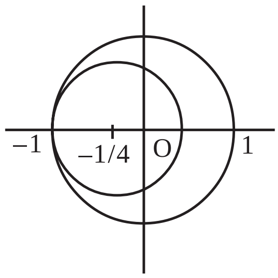


Fig. 4

$$\Psi(z) = \frac{4}{3}\left\{1 - \frac{4}{3}\left(z + \frac{1}{4}\right) + \left(\frac{4}{3}\right)^2\left(z + \frac{1}{4}\right)^2 - \ldots\right\}$$

$$\left|z + \frac{1}{4}\right| < \frac{3}{4} \qquad \text{... (5)}$$

It can be checked easily that $\phi(z)$ and $\Psi(z)$ are direct analytic continuation of each other.

Again in the neighbourhood of z = i/2 we obtain an expansion

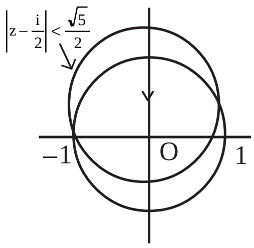


Fig. 5

$$k(z) = \frac{1}{1+i/2}\left[1 - \left(\frac{z - i/2}{1+i/2}\right) + \left(\frac{z - i/2}{1+i/2}\right)^2 - \ldots\right]$$

$$\left|z - \frac{i}{2}\right| < \frac{\sqrt{5}}{2} \qquad \text{... (6)}$$

In performing analytic continuations we notice that there are certain points which always lie on the boundary of domains in which expansions are not valid. These points are nothing but the singularities of the complete analytic function. In example 1.2 these are z = ± i whereas it is z = –1 for example 1.3.

Regular and Singular points

Let f(z) be an analytic function defined in the domain D, bounded by a simple closed curve Γ. A point $\varsigma \in \Gamma$ is called a **regular point** of the function f(z) if there exist a neighbourhood $|z - \varsigma| < \in$ of the point ς and an analytic function $\phi_\varsigma(z)$ such that $\phi_\varsigma(z) = f(z)\ \forall z \in D \cap |z - \varsigma| < \in$.

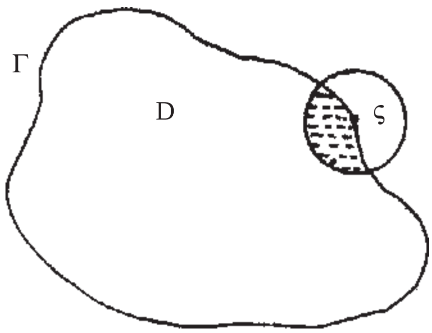


Fig. 6

The boundary point ζ which is not a regular point is called a **singular point** of f(z) i.e., in any neighbourhood of the point ζ, there cannot be any analytic function coinciding with f(z) in the part common to the neighbourhood of ζ and the domain D.

Natural boundary

In examples 1.2 and 1.3 we have encountered with finite number of singular points situated on the boundary of the region of analyticity of the given function. It might happen that the boundary is dense with singular points. In this case analytic continuation across the boundary of the region is not possible. Such a boundary is called a **natural boundary**.

Example 1.4 Test whether analytic continuation of the function $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ is possible outside its circle of convergence.

Solution : Applying the ratio test we find that the given series

$$f(z) = z + z^2 + z^4 + z^8 + \dots \tag{7}$$

converges for $|z| < 1$. The point $z = 1$ is a singular point of f(z) as it is seen for real z that the sum $\sum_{n=0}^{\infty} x^{2^n}$ increases indefinitely as $x \to 1$. Now to test whether the circle of convergence, the unit circle, is a natural boundary we examine the behaviour of the given function at the points.

$$z_{k,s} = e^{\frac{i2\pi}{2^k}s}, \; s = 1, \; 2, \; 3, \; \dots 2^k$$

(k is any natural number). For this sake we consider the points $\tilde{z}_{k.s} = re^{\frac{i2\pi}{2^k}s}$ $0 < r < 1$ and evaluate f(z) at these points.

Then $f(\tilde{z}_{k,s}) = \sum_{n=0}^{k-1} r^{2^n} e^{\frac{i2\pi}{2^k}s.2^n} + \sum_{n=k}^{\infty} r^{2^n} e^{\frac{i2\pi}{2^k}s.2^n}$

and observe that the first term consists of a finite number of terms and hence bounded in absolute value, whereas the second term is absolute value reduces to $\sum_{n=k}^{\infty} r^{2^n}$. Clearly this sum increases indefinitely as $r \to 1$. This shows that the points $z_{k,s}$ (as $\lim_{r \to 1} \tilde{z}_{k,s} = z_{k,s}$ are singular points of the

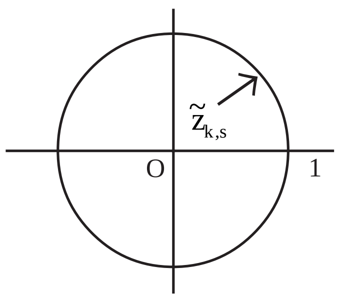


Fig. 7

given function f(z). Now as $k \to \infty$ these points form an everywhere dense set of points on the boundary of the unit circle. Thus analytic continuation outside the circle of convergence of the given function is not possible.

Example 1.5 Show that the function $f(z) = \sum_{n=1}^{\infty} z^{n!}$ has unit circle as its natural boundary.

Theorem 1.2 Every power series has at least one singular point on its circle of convergence.

Proof. Let $f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots$ be any power series with region of convergence $K:|z - z_0| < R$. We shall have to prove there lies at least one singular point on the circle of convergence $\Gamma:|z - z_n| = R$ of the function. Suppose, on the contrary, that every point on Γ are regular points. Let $\varsigma_1, \varsigma_2, \ldots \varsigma_i, \ldots$ be certain number of regular points belonging to Γ and $N(\varsigma_1)$, $N(\varsigma_2)$, ..., $N(\varsigma_i)$... be their neighbourhoods respectively. The points ς_i's are chosen in such a way that $N(\varsigma_i)$ has non null intersection with $N(\varsigma_{i-1})$ and $N(\varsigma_{i+1})$ and the union of these neighbourhoods completely cover the boundary Γ. Let D be the union of K and all these neighbourhoods $N(\varsigma_i)$. D is open since K and every $N(\varsigma_i)$ are open. D is also connected since.

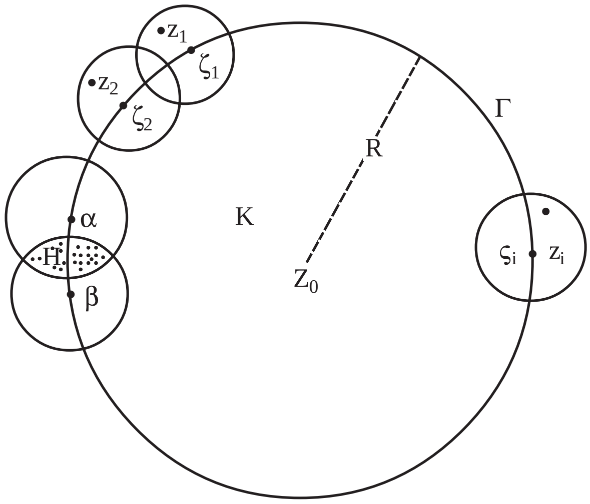


Fig.8

(i) any two points lying in $K \subset D$ can be connected by a straight line segment lying in K, since K is connected.

(ii) one point $z_1 \in N(\varsigma_1)$ and the other $z_2 \in K$ can be connected by two straight line segments $\overline{z_1\zeta_1}$ and $\overline{\zeta_1 z_2}$ lying within $N(\varsigma_1) \cup K \subset D$.

(iii) one point $z_m \in N(\zeta_m)$ and $z_n \in N(\zeta_n)$ can be connected by a curve consisting of $\overline{z_m\zeta_m} + \zeta_m\zeta_n + \overline{\zeta_n z_n} \subset D$ since $\overline{z_m\zeta_m} \subset N(\zeta_m) \subset D$, $\zeta_m\overset{\frown}{\zeta_n} \subset \Gamma \subset D$ and $\overline{\zeta_n z_n} \subset N(\zeta_n) \subset D$.

and finally if two points lie in the same neighbourhood $N(\zeta_i)$ it is always connected by a curve $\gamma \subset N(\zeta_i) \subset D$. Now we introduce an analytic function $\psi(z)$ on the open connected set D which satisfies

$$\psi(z) = \phi_{\varsigma_i}(z),\ z\ \varepsilon\ N(\zeta_i)$$
$$\qquad f(z),\ z\,\varepsilon\,K$$

where $\phi_{\varsigma_i}(z)$ is a direct analytic continuation of f(z) in the neighbourhood $N(\zeta_i)$ of the regular point ζ_i.

We now prove that $\psi(z)$ is well-defined on D. Let α, β be any two points on Γ such that $H = N(\alpha) \cap N(\beta) \neq \phi$ and since α, β are regular points there exist functions $\phi_\alpha(z)$ and $\phi_\beta(z)$ as direct analytic continuations of f(z) in $N(\alpha)$ and $N(\beta)$ respectively i.e.

$$\phi_\alpha(z) = f(z)\ \forall z\ \varepsilon\ N(\alpha) \cap K$$
$$\phi_\beta(z) = f(z)\ \forall z\ \varepsilon\ N(\beta) \cap K$$

so that $\phi_\alpha(z) = \phi_\beta(z) = f(z)\ \forall z \varepsilon G = \left(N(\alpha) \cap K\right) \cap \left(N(\beta) \cap K\right) \subset H$. Now since $\phi_\alpha(z), \phi_\beta(z)$ are analytic in H and G is a part of H, by the uniqueness theorem $\phi_\alpha(z) \equiv \phi_\beta(z)$ $\forall z\ \varepsilon\ H$. As α and β are arbitrary points of Γ we conclude that $\psi(z)$ is a well-defined analytic function on D. Let C be the boundary of D and let $\rho = \overline{z_0\zeta},\ \zeta\ \varepsilon\ C$ be the minimum distance from z_0 to the boundary C of D. Then clearly $\rho > R$ as ς lies outside the circle Γ. Thus we observe that $\psi(z)$ coincides with f(z) on the disc $|z-z_0| < R$. Then it is obvious to conclude that the radius of convergence of the given power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is ρ, not R, which is a contradiction. Hence every point on Γ cannot be regular points, i.e., there must be at least one singular point on Γ.

1.5 Analytic continuation along a curve

Earlier, analytic continuation by power series method, we have extended f(z) to a

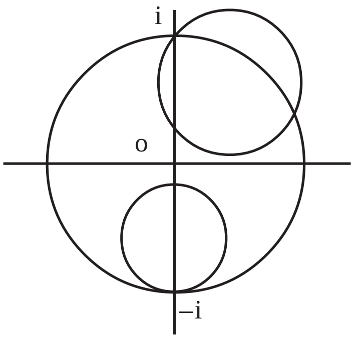


Fig. 9

larger domain considering its power series expansion about a point a from its original circle of convergence with centre at z_0 ($-a \neq z_0$) and radius r. We know, this power series converges in the disc $D_1 : |z - a| < R$, where $R \geq r - |z_0 - a|$ [(see Fig. 9), for example 1.2]. Then it converges to an analytic function g(z) defined on D_1, which is equal to f(z) on $D \cap D_1$.

Analytic continuation along a curve is an extension of this idea to the situation where a curve is covered by an overlapping sequence of discs and an analytic function defined on the first disc, can be extended succesively to each disc in the sequence (see figure 10). We will make this idea more precise after introducing the definition of function element.

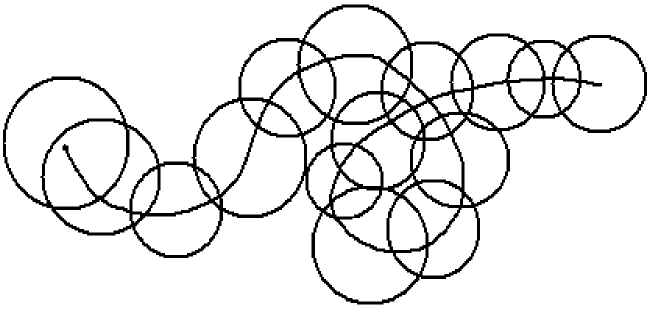


Fig. 10

Definition 1. An ordered pair (f, D), where D is a region and f is an analytic function on D is called a **function element**. We say that it is a function element at z_0 if z_0 belongs to D. Two function elements (ϕ, G) and (ψ, H) are equal if and only if $\phi(z) \equiv \psi(z)$, G = H.

Clearly a function element (f_1, D_1) is a direct analytic continuation of another function element (f_2, D_2) when $D_1 \cap D_2 \neq \phi$ and $f_1 = f_2$ in $D_1 \cap D_2$. In this case the two function elements (f_1, D_1) and (f_2, D_2) are said to be equivalent.

Definition 2. Let $\gamma : [0,1] \to \mathbb{C}$ be a curve and (f_0, D_0) be a function element at $z_0 = \gamma(0)$. Suppose there exists

(i) a partition $0 = t_0 < t_1 < ... < t_n = 1$ of [0, 1] and

(ii) a finite sequence of function elements

(f_0, D_0), (f_1, D_1), ..., (f_n, D_n)

with $\gamma([t_j, t_{j+1}]) \subset D_j$ and (iii) $f_j(z) = f_{j+1}(z)$ on $D_j \cap D_{j+1}$ for j = 0, 1, ... n–1.

Then (f_n, D_n) is called an analytic continuation of (f_0, D_0) along γ. Apparently, it seems that the function element (f_n, D_n) of the above definition, depends on the choice of partition $0 = t_0 < t_1 < ... < t_n = 1$ of [0, 1] and the finite sequence (f_0, D_0), (f_1, D_1), ..., (f_n, D_n) of function elements. It turns out that up to equivalence, it is actually independent of these choices.

Theorem 1.3 Given a curve $\gamma:[0,1]\to\mathbb{C}$ beginning at z_0 and ending at z_n and a function element (f_0, D_0) at z_0, any two analytic continuations of (f_0, D_0) along γ give rise to two function elements at z_n that are direct analytic continuations of each other. [Though the theorem can be proved by taking different partitions of [0, 1] for two different analytic continuations of (f_0, D_0) along γ, here we prove the theorem taking the same partition of [0, 1] for two analytic continuations along γ].

Proof. Let (f_0, F_0), (f_1, F_1), ... (f_n, F_n) and (g_0, G_0), (g_1, G_1), ..., (g_n, G_n) be two analytic continuations of (f_0, D_0) along γ, using the same partition,

$$0 = t_0 < t_1 < ... < t_n = 1$$

where $\gamma(t_j) = z_j$ and $\gamma([t_j, t_{j+1}]) \subset F_j$ and $\gamma([t_j, t_{j+1}]) \subset G_j$ for $j = 0, 1, ..., n$.

By given hypothesis, $(f_0, D_0) = (f_0, F_0) = (g_0, G_0)$. Now we set $E_j = F_j \cap G_j$ for $j = 1, 2, ... n$, and $E_0 = F_0 = G_0$. Then each E_j is a connected open set containing $\gamma(t_j)$ and $\gamma(t_{j+1})$. To prove the theorem we show, by induction, that $f_n = g_n$ on E_n.

We have $f_0 = g_0$ on $E_0 = F_0 = G_0$ by definition. Suppose $j < n$ and $f_j = g_j$ on E_j. But we have

$$f_j = f_{j+1} \quad \text{on } f_j \cap F_{j+1}$$

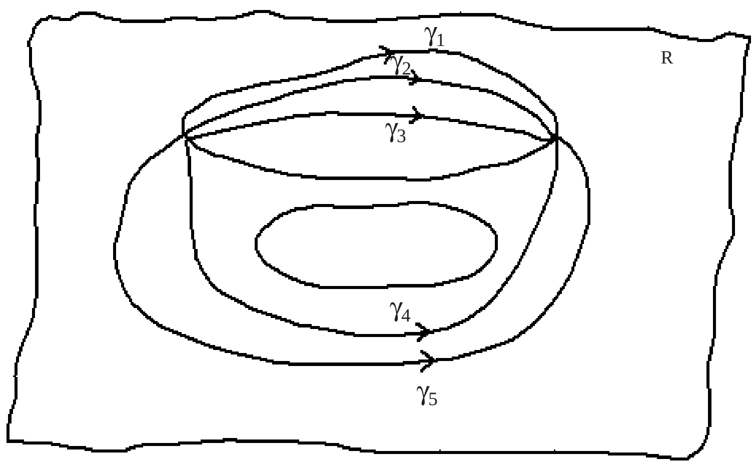
and $$g_j = g_{j+1} \quad \text{on } G_j \cap G_{j+1}$$

and $\gamma(t_{j+1})$ is common to both the open sets $F_j \cap F_{j+1}$ and $G_j \cap G_{j+1}$. So it follows that

$$f_{j+1} = g_{j+1}$$

in a neighbourhood of $\gamma(t_{j+1})$ and hence on E_{j+1} by the uniqueness theorem. By induction the proof is therefore complete.

Homotopic curves. Two arcs γ_1 and γ_2, with common end points, contained in a region R are said to be homotopic if one can be obtained from the other by continuous deformation where the process of continuous deformation must be confined in R.



In the given figure $\{\gamma_1, \gamma_2$ and $\gamma_3\}$ is one set of homotopic curves while $\{\gamma_4, \gamma_5\}$ is the other set. Here no curve of the first set is homotopic to any curve of the second set. These are geometrical interpretations. We now explain such a deformation in an analytical manner.

Let us suppose $\gamma_0 : z = \sigma_0$ (t), $0 \le t \le 1$ and $\gamma_1 : z = \sigma_1$ (t), $0 \le t \le 1$ be two curves, lying in a region R, having common end points a and b i.e., $a = \sigma_0(0) = \sigma_1(0)$ and $b = \sigma_0(1) = \sigma_1(1)$ hold. We say that the curve γ_0 can be continuously deformed into the curve γ_1 keeping the process confined to R, if there exists a function $\sigma(t, s)$ which is continuous in the unit square $I^2 = I \times I$, $I = [0, 1]$ and satisfies the following conditions :

(i) for each fixed s ε [0, 1] the curve $\gamma_s : z = \sigma$ (t, s), $0 \le t \le 1$ lies in R.

(ii) σ (t, 0) = σ_0(t) and σ (t, 1) $\equiv \sigma_1$ (t), $0 \le t \le 1$

(iii) σ (0, s) $\equiv$ a and σ (1, s) $\equiv$ b, $0 \le s \le 1$.

Let α and ς be two points lying in a domain D and suppose that γ_0 and γ_1 are two curves connecting α to ς. Let there exist, as in definition 2, two finite sequences of function elements (f_0, G_0), (f_1, G_1) ..., (f_n, G_n) and (g_0, H_0), (g_1, H_1), ..., (g_m, H_m) along the curves γ_0 and γ_1 respectively. We also suppose that the function elements (f_0, G_0) and (g_0, H_0) at the point α are equivalent. Then a question arises whether the function elements (f_n, G_n) and (g_m, H_m) at the point ς are also equivalent? If γ_0 and γ_1 are the same curve the Th. 1.3 confirms the answer for equivalence. However, if γ_0 and γ_1 are distinct there is no definite answer. The reason behind this is the fact that the regions enclosed by the curves γ_0 and γ_1 may contain points at which we can not find any function element that can be included in the sequence of function elements from the point α to ς along any curve passing through these points. Here we discuss a few problems highlighting these facts :

Example 1.6 Let $Q_1 = \{z \;\varepsilon\; \mathbb{C} \mid \text{Re } z > 0, \text{Im } z > 0\}$ denote the first quadrant and set f(z) = log z for all z ε Q_1

Show that, if g_1 is the analytic continuation to $\mathbb{C}\backslash(-\infty, 0]$ of f and g_2 is the analytic continuation to $\mathbb{C}\backslash[0, \infty)$ of f, then $g_1 \neq g_2$ throughout the third quadrant, $Q_3 = \{z \;\varepsilon\; \mathbb{C} \mid \text{Re } z < 0, \text{Im} z < 0\}$.

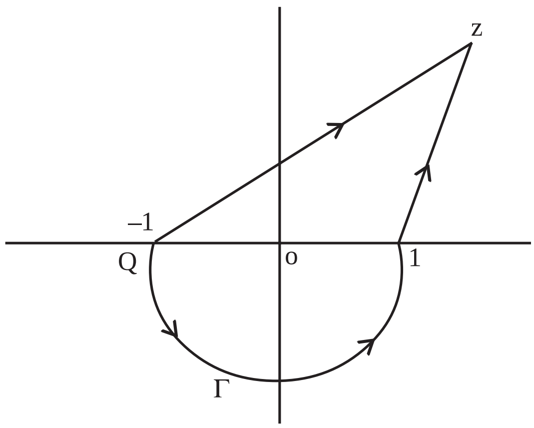


Fig. 10

Proof. Clearly, g_1 is the principal branch of log z throughout $\mathbb{C}\backslash(-\infty, 0]$

by the uniqueness theorem. That is

$$g_1(z) = \int_{[1,z]} \frac{d\varsigma}{\varsigma}$$

for all z barring the negative real axis including origin. We define

(i) $g_2(z) = \int_{[-1,z]} \frac{d\varsigma}{\varsigma} + i\pi$ for all z ε $\mathbb{C} \setminus [0, \infty]$

and show that

(ii) $g_2(z) = g_1(z) + 2\pi i$ for all z ε Q_3.

Let γ be the closed curve (see figure) consisting of the line segments [1, z], [z, –1] and a semi-circular path Γ with centre at the origin and radius 1, where z is any point in Q_1.

Now we wish to calculate

$$\int_\gamma \frac{d\varsigma}{\varsigma}$$

By Cauchy's Residue Theorem, it is equal to 2πi origin is the only pole inside γ). So breaking up the contour γ, we can equate

$$2\pi i = \int_{[1,z]} \frac{d\varsigma}{\varsigma} + \int_{[z-1]} \frac{d\varsigma}{\varsigma} + \int_\Gamma \frac{d\varsigma}{\varsigma}$$

$$= g_1(z) - \int_{[-1,z]} \frac{d\varsigma}{\varsigma} + i\pi$$

i.e., $$g_1(z) - \int_{[-1,z]} \frac{d\varsigma}{\varsigma} + i\pi = g_2(z)$$

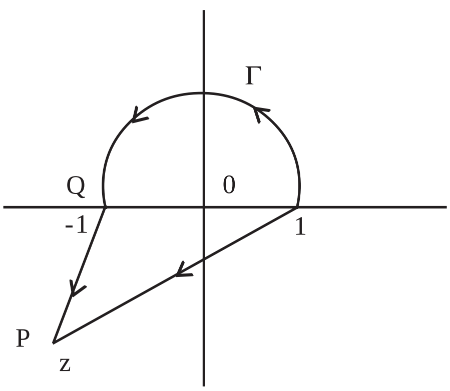


Fig. 11

Hence $g_2(z) = g_1(z) = \log z$ for all z ε Q_1, that is, the mapping g_2 defined in (i) is the unique analytic continuation of f to $\mathbb{C} \setminus [0, \infty)$.

To prove (ii) Let z ε Q_3 and γ be the curve joining the line segments [–1, z], [z, +1] and a unit semi-circular path Γ in the upper half plane. Thus

$$2\pi i = \int_\gamma \frac{d\varsigma}{\varsigma} = \int_\Gamma \frac{d\varsigma}{\varsigma} + \int_{[-1,z]} \frac{d\varsigma}{\varsigma} + \int_{[z-1]} \frac{d\varsigma}{\varsigma}$$

$$= \pi i + \int_{[-1,z]} \frac{d\varsigma}{\varsigma} - g_1(z)$$

i.e., $g_2(z) = g_1(z) + 2\pi i$ for all $z \varepsilon Q_3$.

Remark : The preceding example presents the following observation : If γ_1 and γ_2 be the two curves joining z_0 and ς, (f_0, D_0) be a function element at z_0, then the resulting function elements of (f_0, D_0) along the curves γ_1 and γ_2 at ς may not be direct analytic continuations of each other. We shall now discuss for what reasons such type of situation occurs.

1.6 Multi-valued Functions and Analytic continuation

When we define both real and complex functions we always keep in mind that for each value of the independent variables the value of the function must be unique. For example, even Cauchy's theorem is based on the assumption that a function can be defined uniquely in the region under consideration. All the same, multivaluedness often arises out of necessity in the actual construction of functions, the simplest example is perhaps the logarithm :

In section 5.2 [14] we showed that if z is a non zero complex number, then the equation $z = e^{\omega}$ has infinitely many solutions. Since the function $f(w) = e^{\omega}$ is a many-to-one mapping, its inverse (the logarithm) is multi-valued.

Definition 3 : [Multi-valued logarithm] : For $z \neq 0$, we define the function log z as the inverse of the exponential function; that is,

$$\log z = \omega \text{ if and only if } z = e^{\omega} \quad (8)$$

If we go through the same steps as we did to obtain (5.5) [14], we find that, for any complex number $z \neq 0$, the solutions ω to equation (8) take the form

$$\omega = \log z = \log |z| + i\theta, \text{ for } z \neq 0 \quad (9)$$

where $\theta \varepsilon \arg z$ and $\log |z|$ denotes the natural logarithm of the positive number $|z|$. Because arg z is the set $\arg z = \text{Arg } z + 2n\pi$, where n is an integer, we can express the set of values comprising log z as

$$\log z = \log |z| + i\,(\text{Arg } z + 2n\pi), \text{ where } n = \text{integer} \quad (10)$$

or $$\log z = \log |z| + i \arg z \text{ for } z \neq 0, \quad (11)$$

where it is understood that the identity (11) refers to the same set of numbers given in identity (10).

We call any one of the values given in identities (10) or (11) a logarithm of z. Notice that the different values of log z all have the same real part and that their imaginary parts differ by the amount $2n\pi$, where n is an integer. Regarding analytic continuation, we treat log z for complex valued z as the extension of log x from positive real domain to complex domain. Consider the Taylor series expansion of log x :

$$\log x = \log\{1 + (x - 1)\} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x - 1)^n, 0 < x < 2 \tag{12}$$

We take this series for complex valued z and write

$$f_0(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z - 1)^n \tag{13}$$

which converges in the disc K_0 : $|z-1| < 1$ so that $f_0(x) = \log x$ for $0 < x < 2$. Thus $f_0(z)$ and log x are direct analytic continuations of each other.

Our object is to specify the curves along which the analytic continuation of the function element (f_0, K_0) is possible. For this purpose it is advantageous to apply the integral representation.

$$\log x = \int_1^x \frac{ds}{s}, 0 < x < \infty \tag{14}$$

Lemma 1.1. The following formula

$$f_0(z) = \int_1^z \frac{d\varsigma}{\varsigma} \tag{15}$$

holds for z ε K_0 where the integral is taken along any path lying completely within K_0.

Proof. The function $f_0(z)$ given by (13) is regular in K_0 and following Theoren 3.2[14] the integral on the r.h.s of (15) is also regular in K_0. But we see that this integral coincides with log x in (14) for $0 < x < 2$. By the uniqueness theorem.

$$f_0(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z - 1)^n = \int_1^z \frac{d\varsigma}{\varsigma},\ z \,\varepsilon\, K_0.$$

In continuing $f_0(z)$ analytically to an arbitrary point ω we isolate a single-valued piece of log z, as we shall do later for other multivalued functions, called a branch of the function. The standard way to isolate **single-valued branches** is by the use of branch cuts to different branches. For log z the question of multivaluedness arises when z goes around the origin, as a result argument changes by 2π. Such a point is called a **branch point.** If we do not allow the paths to travel around a branch point of a multi-valued function then certainly we would not face varied values at a point lying in the domain of definition of the function.

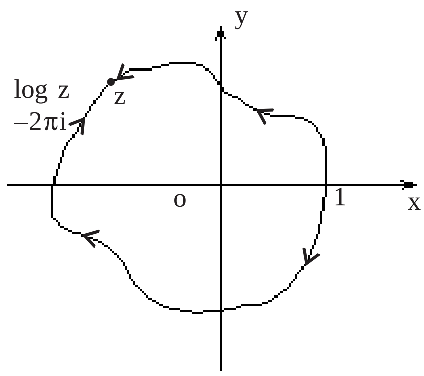


Fig. 12

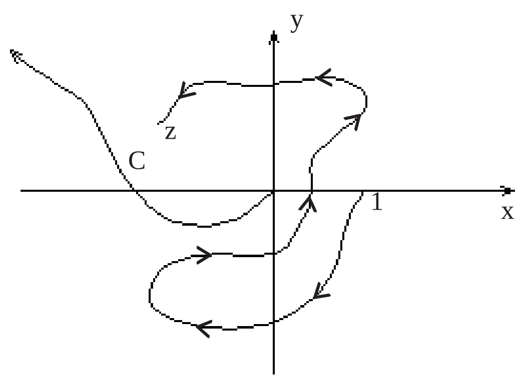


Fig. 13

Let C be any simple curve from 0 to ∞, so that z cannot go around the origin crossing C.

The above consideration shows that if analytic continuation along a given curve Γ is possible, then one can get from a function element at the initial point of the curve another function element at the terminal point of the curve by a finite number of applications of direct analytic continuation. If there is no function element at the initial point of Γ that can be continued along Γ, then there exists a definite point on the curve Γ which is a singular point at which the process of analytic continuation must stop.

The following question immediately arises : if ω is some non-singular point outside the disc D_0, then there may two or more chains of function elements which eventually continue analytically f_0 (z) onto a disc D containing ω. For example, let (f_j, D_j) be the function element of one chain and (f_k, D_k) be the function element of a different chain and that $\omega \in D_j \cap D_k$; will then $f_j(z) = f_k(z)\ \forall\ z \in D$?

The Monodromy Theorem

The above question is answered by the Monodromy theorem, which, simply stated, is : if there are no singular points in between the two paths of analytic continuation, then the result of analytic continuation is the same for each path. Another way of stating the theorem is :

Theorem 1.4 [Monodromy Theorem] Let (f_0, D_0) be a function element at z_0 and R be a simply connected region containing D_0, ς be any point lying in R. We suppose

(i) (f_0, D_0) can be analytically continued along every curve in R.

(ii) γ_0 and γ_1 are homotopic curves from z_0 to ς.

Then the continuations of the function element (f_0, D_0) along γ_0 and γ_1 at ς are equivalent.

Proof. A homotopy from γ_0 to γ_1 determines a continuous one parameter family of curves $\{\gamma_s\}$, $0 \le s \le 1$ from z_0 to ς given by the equations $z = \sigma_s(t)$, $0 \le t \le 1$.

By hypothesis, the function element (f_0, D_0) has an analytic continuation along each of the curves, γ_s. Denote the terminal function element at ς for the continuation along γ_s by ϕ_s. We claim that, for each $k \varepsilon [0, 1]$, there is a $\delta > 0$ such that ϕ_s is equivalent to ϕ_k whenever $|s–k| < \delta$.

Let $0 = t_0 < t_1 < .... < t_n = 1$ be a partition and (f_0, D_0), (f_1, D_1), ..., (f_n, D_n) be a finite sequence of function elements defining $\phi_k = (f_n, D_n)$ as the terminal function element at ς for the analytic continuation of (f_0, D_0) along γ_k. Then

$$E_j = \sigma_k\ ([t_j,\ t_{j+1}]) \subset D_j \text{ for } j = 0,\ 1,\ ...,\ n–1$$

For each $j = 0, 1, ... n–1$, let ε_j be the minimum distance from the compact set E_j to the boundary of the D_j. If $|\sigma_s(t)–\sigma_k(t)| < \varepsilon_j$, $t \varepsilon [0, 1]$, then it will also be true that $\sigma_s\ ([t_j,\ t_{j+1}]) \subset D_j$. Thus, if $\varepsilon = \min \{\varepsilon_0, \varepsilon_1, ... \varepsilon_{n–1}\}$ and we choose $\delta > 0$ such that $|\sigma_s(t) – \sigma_k(t)| < \varepsilon$ whenever $|s–k| < \delta$, then for each s with $|s–k| < \delta$, the partition $0 = t_0 < t_1 < ... < t_n = 1$ and sequence of function elements (f_0, D_0), (f_1, D_1), ..., (f_n, D_n) also defines (f_n, D_n) as the terminal function element at ς for the analytic continuation of (f_0, D_0) along γ_s. Since, by the previous theorem 1.3, any other continuation of (f_0, D_0) along γ_s results function element equivalent to this one, we conclude that ϕ_k is equivalent to ϕ_s. This proves that ϕ_s is equivalent to ϕ_k whenever $|s–k| < \delta$.

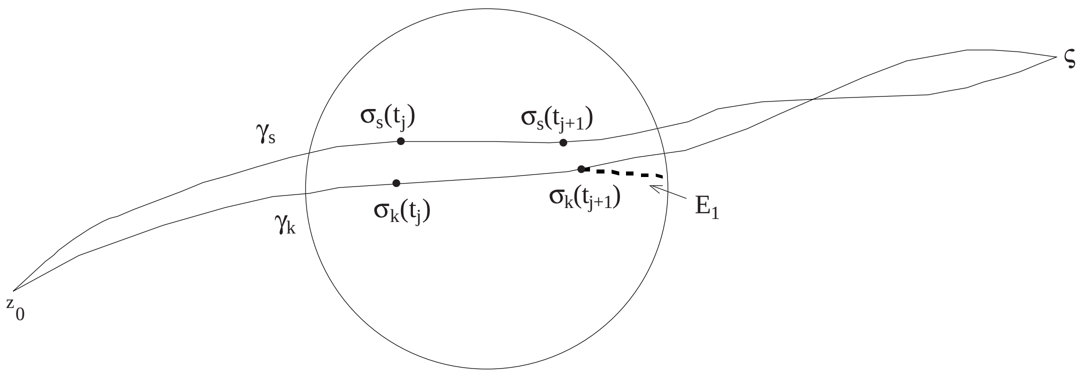


Fig. 14

This means that for every $s \varepsilon I = [0, 1]$ there is a positive $\delta(s)$ such that if s lies in the interval $I_s = (s–\delta(s), s + \delta(s))$, then the analytic continuation of $f_0(z)$ along all such curves γ_s, result equivalent function elements at the point ς. Now by the Heine-Borel theorem, we can always choose a finite number of intervals I_{sj}, $0 = s_0 < s_1 < .... < s_n = 1$ that cover the segment I and are such that the intervals I_{sj} and

$I_{s_{j+1}}$, $0 \le j \le n-1$ have a non-empty intersection. Then, if $s \varepsilon I_{s_0} \cap I_{s_1}$, the analytic continuation of $f_0(z)$ result equivalent function elements at the point ς. The same is true for $s \varepsilon I_{s_1} \cap I_{s_2}$ and so on. Continuing in this way we observe that the analytic continuation of the function element (f_0, D_0) along all the curves γ_s, $0 \le s \le 1$ produce equivalent function elements at the point ς. This completes the proof of the theorem.

The above theorem leads us to the following most important corollary.

Corollary. Let R be a simply connected region and

(i) (f_0, D_0) be a function element at z_0 belonging to R

(ii) (f_0, D_0) admit analytic continuation along every curve in R.

Then there is a function F which is analytic on R and coincides with f_0 on D_0.

Proof. Let z_1 be a point in R. Then, since R is simply connected any two curves from z_0, to z_1 are homotopic in R. The Monodromy theorem implies that any two terminal function elements of analytic continuations of (f_0, D_0) along curves from z_0 to z_1 in R will be equivalent and hence, will determine a function F_1 analytic in some neighbourhood of z_1, say Q_1.

Clearly, $F_1(z) = f_0(z)$ on D_0, $F_1(z) = f_1(z)$ on D_1, ..., etc for the continuation along the curve γ_1 from z_0 to z_1.

Again let z_2 be a point in R, and γ_2 be a curve in R joining z_0 to z_2 and let (g_n, E_n) be the function element at z_2 continuing along the curve γ_2 with $f_0 = g_0$ on $D_0 = E_0$. We simply join z_2 to z_1 by a curve γ and claim that continuation of (F_1, Q_1), along the curve γ to z_2, will be equivalent to (g_n, E_n) (since the curves $\gamma_1 \cup \gamma$ and γ_2 are homotopic), which gives rise to the fact that there is a function F_2 analytic in some neighbourhood of z_2, say Q_2, which coincides with F_1 On Q_1.

Clearly, $F_2(z)$ possesses larger domain of analyticity than $F_1(z)$. Proceeding in this way finite number of times we can achieve a function F analytic throughout the region R.

Unit 2 ❑ Harmonic Functions

Structure

2.0 Objectives

In this chapter we shall mainly study harmonic functions and their basic properties. Gauss' mean value theorem, Poisson's integral formula, Dirichlet's problem for a disc and Harnack inequality for harmonic functions will be discussed. Subharmonic and super harmonic functions will be explained through examples.

2.1 Harmonic Function

A function u(x, y) of two real variables x and y defined in an open set D is said to be harmonic in D if it has continuous derivatives of the second order and satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (16)$$

known as Laplace's equation.

The differential operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called the Laplacian and is denoted by ∇^2.

We introduce the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \qquad (17)$$

in order to achieve a condition equivalent to (16) for f(z). If we write

$$x = \frac{1}{2}(z + \bar{z}) \text{ and } y = \frac{1}{2i}(z - \bar{z}) \tag{18}$$

then

$$\left.\begin{aligned} \frac{\partial f}{\partial z} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y} \end{aligned}\right\} \text{(19a–b)}$$

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial x}{\partial z} + \frac{\partial^2}{\partial x \partial y}\cdot\frac{\partial y}{\partial z}\right] - \frac{1}{2i}\left[\frac{\partial^2 f}{\partial x \partial y}\cdot\frac{\partial x}{\partial z} + \frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial y}{\partial z}\right]$$

$$= \frac{1}{4} f_{xx} + \frac{1}{4i} f_{xy} - \frac{1}{4i} f_{xy} + \frac{1}{4} f_{yy} = \frac{1}{4}\left(f_{xx} + f_{yy}\right)$$

and consequently the condition equivalent to (16) is

$$\nabla^2 f = 4\frac{\partial^2 f}{\partial z \partial \bar{z}} \tag{20}$$

A function f(z) is said to be harmonic in D if f has continuous second derivatives in D and satisfies

$$\nabla^2 f = 0, \ \forall z \, \varepsilon \, D \tag{21}$$

Result 1 : If f = u + iv is analytic in a domain D, then $\frac{\partial f}{\partial \bar{z}} = 0, \ \forall z \, \varepsilon \, D$.

Proof : u and v satisfy the Cauchy-Riemann equations and using (19b) we have,

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(u_x + iv_x) - \frac{1}{2i}(u_y + iv_y)$$

$$= \frac{1}{2}(u_x + iv_x) - \frac{1}{2i}(-v_x + iu_x), \text{ using C–R equations}$$

$$= 0$$

Result 2 : The real and imaginary parts of an analytic function are harmonic.

Proof : Let f = u + iv be analytic in a domain D. By Cauchy-Riemann equations

$$u_x = v_y \text{ and } u_y = -v_x$$

i.e. $u_{xx} = v_{xy}$ and $u_{yy} = -v_{xy}$ [since $v_{xy} = v_{yx}$, partial derivatives being continuous] and on addition it proves that u is harmonic in D. Likewise v is also harmonic in D.

Harmonic conjugates : Let u (x, y) and v(x, y) be two harmonic functions in a domain $D \subseteq \mathbb{C}$.

If they satisfy the Cauchy-Riemann equations :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \text{ in D, then}$$

we say that v is a harmonic conjugate of u. It follows that f(z) = u(x, y) + i v (x, y) is analytic in a domain D if and only if v(x, y) is a harmonic conjugate of u(x, y) in D.

Remark : We know that the real part as well as the imaginary part of an analytic function are harmonic. Now the questions arise :

1. Can any real harmonic function be the real part of an analytic function?

2. Whether every real harmonic function has a harmonic conjugate?

Existence of Harmonic conjugates

Theorem 2.1 Let u(x, y) be a real-valued harmonic function in a simply connected domain $D \subseteq \mathbb{C}$. Then there is an analytic function f in D such that u = Re f (or, equivalently there is a function v, a harmonic conjugate of u) which is unique to within addition of an arbitrary real constant.

Proof. Since the function u(x, y) is harmonic in a simply connected domain D, we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

which can be rewritten as

$\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$, where $-\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial x}$ are given functions with continuous first partial derivatives. This implies that

$$-\left(\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy$$

is exact. So there is a single-valued function v(x, y) which is unique to within an additive arbitrary constant, i.e.

$$v(x, y) = \int_{(x_0, y_0)}^{(x,y)} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy + K \qquad (22)$$

$K \equiv$ real constant,

where (x_0, y_0) is an initial point and (x, y) is any variable point lying in D and the integral on the curve connecting (x_0, y_0) to (x, y) is path independent.

From (22) we find that

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \ \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x},$$

which in turn ensures that v(x, y) is harmonic in D and harmonic conjugate to u(x, y) i.e. f = u + iv forms an analytic function in D.

Observation : If D is multiply connected then the integral in (22) may take different values for different paths connecting (x_0, y_0), to (x, y) giving v(x, y) as a multi-valued function, unless the paths are restricted to a simply connected sub domain contained in D.

Example 1. Let D be the whole plane cut along the negative real axis including the origin (y = 0, x ≤ 0). Show that u(x, y) = sin x cosh y is harmonic in D, and find its harmonic conjugate. Also find the corresponding analytic function.

Solution : Here u(x, y) possesses continuous second order partial derivatives in D and also satisfies the Laplace equation : $u_{xx} + u_{yy} = 0$. Hence u(x, y) is harmonic in D.

Let v(x, y) be its harmonic conjugate. Then according to the formula (22), we have

$$v(x, y) = \int_{(1,0)}^{(x,y)} \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) + K, \ K \equiv \text{ real constant,}$$

where M(1, 0) is the initial point.

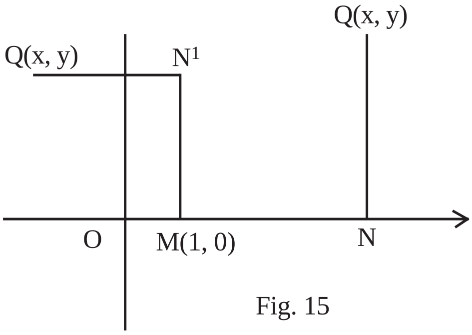


Fig. 15

Here, u(x, y) = sin x cosh y

u_x = cos x cosh y

u_y = sin x sinh y

Now let the point Q(x, y) lie in the 1st quadrant of the right-half plane. Then integrating along MNQ, we find that

$$v(x, y) = \int_{MN} -\frac{\partial u}{\partial y} dx + \int_{NQ} -\frac{\partial u}{\partial x} dy + K_1$$

$$= -\int_1^x \sin x \sinh o \, dx + \int_0^y \cos x \cosh y \, dy + K_1$$

$$= \cos x \sinh y + K_1$$

Again, if the point (x, y) lies in the 2nd quadrant of the left-half plane, then we obtain

$$v(x, y) = \int_{MN^1} \frac{\partial u}{\partial x} dy + \int_{N^1Q} -\frac{\partial u}{\partial y} dx + K_2$$

$$= \int_0^y \cos 1 \cosh y \, dy + \int_1^x - \sin x \sinh y \, dx + K_2$$

$$= \cos 1 \sinh y + \cos x \sinh y - \cos 1 \sinh y + K_2$$

$$= \cos x \sinh y + K_2$$

The expression for v(x, y) in both the cases turns out to be the same apart from an additive constant. It results from the fact that the two paths in determining the integral lie in a simply connected domain. Thus, $v(x, y) = \cos x \sinh y + K$ at all points of D. Therefore, an analytic function with the given real part will be of the form

$$\begin{aligned} f(z) &= \sin x \cosh y + i \cos x \sinh y + iK, \; K \equiv \text{real constant} \\ &= \sin(x + iy) + iK \\ &= \sin z + iK \end{aligned}$$

As for uniqueness, if two analytic functions in D have the same real part, then their difference has derivative zero, by the Cauchy-Riemann equations. In that case the functions differ by a constant.

2.2 Gauss' Mean Value Theorem for harmonic functions

Let $u(z) = u(x, y)$, $z = x + iy$, be harmonic in the disk $K : |z - z_0| < R$ and continuous on the closed disk $\overline{K}$. Then

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{i\theta})\, d\theta \tag{23}$$

Proof. Let f(z) be an analytic function defined in K such that Re f (z) = u(z). It follows from Cauchy's integral formula that

$$f(z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{z - z_0}\, dz, \;\; 0 < r < R$$

using the parametric form of the circle $|z - z_0| = r$.

$z = z_0 + re^{i\theta}, \; 0 \le \theta \le 2\pi,$ so that $dz = ire^{i\theta}d\theta.$ The integral then gives

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta, \;\; 0 < r < R$$

Equating the real parts, we obtain

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta$$

whence taking the limit $r \to R$, we obtain the desired result (23)

2.3 Inverse point of a given point with respect to a circle

Let $\gamma : |z - \alpha| = R$ and z_0 be a given point. Let z_1 be another point on the radius through z_0 such that $|z_0 - \alpha|\,|z_1 - \alpha| = R^2$. Then either of the points z_0 and z_1 is called the inverse point of the other with respect to γ. The centre of the circle γ is called the centre of inversion.

It follows from the definition that (i) if z_0 lies inside γ, then z_1 must lie outside

γ, (ii) if z_0 lies on γ, then z_1 must also lie on γ and it coincides with z_0, (iii) if z_0 lies outside γ, then z_1 must lie inside γ.

Every point, except the centre of the circle, on the plane has a unique inverse point with respect to the circle. We associate the point at infinity to the inverse point of the centre.

Result : Let $\gamma : |z| = R$ and z_0 be a given point. Then the inverse point of z_0 with respect to γ is given by $\dfrac{R^2}{\bar{z}_0}$.

Proof : Let $z_0 = re^{i\theta}$. Then its inverse point with respect to γ is given by $z_1 = r_1e^{i\theta}$, where $rr_1 = R^2$. Hence $r_1 = \dfrac{R^2}{r}$ and so

$$z_1 = \frac{R^2}{r} \cdot e^{i\theta} = \frac{R^2}{re^{-i\theta}} = \frac{R^2}{\bar{z}_0}$$

Poisson's integral formula : Theorem : Let u(x, y) be a harmonic function in a simply connected region D and $\gamma : |\varsigma| = R$ be a circle contained in D. Then for any $z = re^{i\theta}$, $r < R$, u can be written as $u(r, \theta) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \dfrac{(R^2 - r^2).\, u(R, \phi)d\phi}{R^2 + r^2 - 2Rr\cos(\phi - \theta)}$, where $Re^{i\phi}$ is a point on γ.

Proof : Since u(x, y) is harmonic in D, there exists a conjugate harmonic function v(x, y) in D so that f(z) = u(x, y) + iv(x, y) is analytic in D. Then f(z) is analytic within and on γ and so for any z within γ, by Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\varsigma)}{\varsigma - z}\, d\varsigma \tag{24}$$

The inverse point of z with respect to γ lies outside γ and is given by $\dfrac{R^2}{\bar{z}}$. Hence by Cauchy-Goursat theorem,

$$0 = \frac{1}{2\pi i}\int_\gamma \frac{f(\varsigma)}{\varsigma - \dfrac{R^2}{\bar{z}}}\, d\varsigma \tag{25}$$

Subtracting (25) from (24) we get,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \left\{ \frac{1}{\varsigma - z} - \frac{1}{\varsigma - \dfrac{R^2}{\bar{z}}} \right\} f(\varsigma)d\varsigma$$

$$= \frac{1}{2\pi i}\int_{\gamma} \frac{\left(z - R^2/\bar{z}\right) f(\varsigma)\, d\varsigma}{(\varsigma - z)\left(\varsigma - \frac{R^2}{\bar{z}}\right)} \tag{26}$$

Let $\varsigma = Re^{i\phi}$. Also, $\bar{z} = re^{-i\theta}$. Then (26) becomes

$$f(re^{i\theta}) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\left(re^{i\theta} - \frac{R^2}{r} e^{i\phi}\right) f(Re^{i\theta})\, i\, Re^{i\theta}\, d\phi}{(Re^{i\theta} - re^{i\theta})\left(Re^{i\phi} - \frac{R^2}{r} e^{i\theta}\right)}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2 - R^2)e^{i(\phi+\theta)} f(Re^{i\phi})\, d\phi}{(Re^{i\phi} - re^{i\theta})(re^{i\phi} - Re^{i\theta})}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{i\phi})\, d\phi}{(Re^{i\phi} - re^{i\theta})(Re^{-i\phi} - re^{-i\theta})}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{i\phi})\, d\phi}{R^2 + r^2 - 2Rr\cos(\phi - \theta)} \tag{27}$$

Let $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$. Then (27) becomes

$$u(r,\theta) + iv(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\{u(R,\phi) + iv(R,\phi)\}}{R^2 + r^2 - 2Rr\cos(\phi - \theta)}\, d\phi \tag{28}$$

Equating real parts in (28) we get,

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2) u(R,\phi)}{R^2 + r^2 - 2Rr\cos(\phi - \theta)}\, d\phi \tag{29}$$

Formula (29) is known as Poisson's integral formula.

Note : Let $\frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\phi - \theta)} = P(R, r, \phi - \theta)$. Then,

the function $P(R, r, \phi - \theta)$ is called the Poisson Kernel. Hence we can write (29) in the form

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} P(R, r, \phi - \phi) u(R,\phi)\, d\phi \tag{30}$$

We can also get a formula similar to (29) for the imaginary part of f(z) by equating the imaginary part in (28). The corresponding formula is

$$v(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)v(R,\phi)d\phi}{R^2 + r^2 - 2Rr\cos(\phi - \theta)} = \frac{1}{2\pi}\int_0^{2\pi} P(R,r,\phi-\theta)v(R,\phi)\,d\phi \qquad (31)$$

Remark : Cauchy's integral formula expresses the values of an analytic function inside a circle in terms of its values on the boundary of the circle whereas Poisson's integral formula expresses the values of a harmonic function inside a circle in terms of its values on the boundary of the circle.

Result 3. $\frac{1}{2\pi}\int_0^{2\pi} P(R,r,\phi-\theta)\,d\phi = 1.$

Proof : By Poisson's integral formula we have,

$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} P(R,r,\phi-\theta)\,u(R,\phi)d\phi$ Taking $u(r, \theta) \equiv 1$ we get,

$$\frac{1}{2\pi}\int_0^{2\pi} P(R,r,\phi-\theta)\,d\phi = 1$$

Result 4. $P(R,r,\phi-\theta) = \mathrm{Re}\left(\frac{\varsigma + z}{\varsigma - z}\right)$

Proof : Let $\varsigma = Re^{i\phi}$, $z = re^{i\theta}$, $r < R$. Then,

$$\frac{\varsigma + z}{\varsigma - z} = \frac{Re^{i\phi} + re^{i\theta}}{Re^{i\phi} - re^{i\theta}} = \frac{(R\cos\phi + r\cos\theta) + i(R\sin\phi + r\sin\theta)}{(R\cos\phi - r\cos\theta) + i(R\sin\phi - r\sin\theta)}$$

$$= \frac{\{(R\cos\phi + r\cos\theta) + i(R\sin\phi + r\sin\theta)\}\{(R\cos\phi - r\cos\theta) - i(R\sin\phi - r\sin\theta)\}}{(R\cos\phi - r\cos\theta)^2 + (R\sin\phi - r\sin\theta)^2}$$

Simplifying we get, $\mathrm{Re}\left(\frac{\varsigma + z}{\varsigma - z}\right) = \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\phi - \theta)} = P(R,r,\phi-\theta).$

Result 5. Poisson Kernel $P(R, r, \phi - \theta)$ is harmonic in $|z| < R$.

Proof : Let $f(z) = \frac{\varsigma + z}{\varsigma - z}$. Then f(z) is analytic in $|z| < R$. By result 4, $P(R, r\ \phi - \theta) = \mathrm{Re}\ f(z)$. Hence the Poisson Kernel is the real part of an analytic function. Hence $P(R, r, \phi - \theta)$ is harmonic in $|z| < R$.

Note : We can easily show that $\frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\phi - \theta)} = \frac{R^2 - |z|^2}{|Re^{i\phi} - z|^2}$

where $z = re^{i\theta}$, $r < R$. Hence $\text{Re}\left(\frac{\varsigma + z}{\varsigma - z}\right) = \frac{R^2 - |z|^2}{|Re^{i\phi} - z|^2}$ and Poisson's integral formula (29) can be written as

$$u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - |z|^2}{|Re^{i\phi} - z|^2} u(R, \phi)\, d\phi \tag{32}$$

The function $\frac{R^2 - |z|^2}{|Re^{i\phi} - z|^2}$ is the Poisson Kernel.

Theorem 2.2 Let $u(x, y) \neq$ constant be harmonic on a simply connected domain D. Then $u(x, y)$ has neither a maximum nor a minimum at any point of D.

Proof. Let $z_0 = x_0 + iy_0$ be an arbitrary point of D. Then following theorem 2.1 there is an analytic function $f(z)$ in a neighbourhood $N(z_0)$ of z_0 such that $\text{Re } f = u$. Then

$$g(z) = e^{f(z)}$$

is analytic on $N(z_0)$ and not equal to constant since $u(x, y) \neq$ constant and

$$|g(z)| = e^{u(x,y)}$$

Again exponential function is strictly increasing, so a maximum for u at (x_0, y_0) is also a maximum for e^u, and hence also a maximum of $|e^f|$ i.e. of $|g(z)|$ at z_0. The function $u(x, y)$ cannot have a maximum at (x_0, y_0), since otherwise $|g(z)|$ would have a maximum at z_0, thereby contradicting the maximum modulus principle. Likewise, following the minimum modulus principle $|g(z)|$ cannot have a minimum value at z_0 since $|g(z)| \neq 0$ on D. Therefore $u(x, y)$ cannot possess minimum value at (x_0, y_0).

Corollary. Let $u(x, y)$ be harmonic on a domain D and continuous on $\overline{D}$. Then $u(x, y)$ attains its maximum and its minimum on the boundary of D.

Proof. Since $u(x, y)$ is continuous on the compact set $\overline{D}$, it attains both its maximum and its minimum on $\overline{D}$, but $u(x, y)$ cannot possess a maximum or a minimum at a point of D. Therefore the corollary follows.

Example 2. Given $u(x, y)$ harmonic in the disk $|z| < R$ and $A(r_j)$ its maximum value on the circle $|z| = r_j$, $r_j < R$, $j = 1, 2, 3$. Prove that

$$A(r_2) \leq \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1} A(r_3) + \frac{\log r_3 - \log r_2}{\log r_3 - \log r_1} A(r_1)$$

for $\quad 0 < r_1 < r_2 < r_3 < R$.

Solution. Since $u(x, y)$ is harmonic in $|z| < R$, $u(x, y) + \alpha \log r$, $r = \sqrt{x^2 + y^2}$, $\alpha \equiv$ a real constant to be fixed later, is also harmonic in the annulus $r_1 \leq |z| \leq r_3$. Hence its maximum is attained on the boundary of the annulus i.e. on $|z| = r_1$ or, $|z| = r_3$ or, on both. Either $A(r_1) + \alpha \log r_1$ or, $A(r_3) + \alpha \log r_3$ is maximum. We define α so that

$$A(r_1) + \alpha \log r_1 = A(r_3) + \alpha \log r_3$$

or, $$\alpha = \frac{A(r_1) - A(r_3)}{\log r_3 - \log r_1}$$

The circle $|z| = r_2$ lies inside the annulus $r_1 \le |z| \le r_3$ and according to corollary of the theorem 2.2 regarding maximum value of the harmonic function $u(x, y) + \alpha \log r$ we have

$$A(r_2) + \alpha \log r_2 \le A(r_3) + \alpha \log r_3$$

or, $$A(r_2) \le A(r_3) + \alpha(\log r_3 - \log r_2)$$

$$= A(r_3) + \frac{A(r_1) - A(r_3)}{\log r_3 - \log r_1}(\log r_3 - \log r_2)$$

$$= \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1} A(r_3) + \frac{\log r_3 - \log r_2}{\log r_3 - \log r_1} A(r_1)$$

2.4 The Dirichlet Problem

Let D be a domain with boundary Γ and let $\bigcup(x, y)$ be a continuous real function defined on Γ. The Dirichlet problem is to find a function $u(x, y)$, harmonic on D and continuous on $\overline{D}$, which coincides with $\bigcup(x, y)$ at every point of Γ.

Existence of a solution of Dirichlet's problem for a disc

Theorem 2.3 Let D be the disc $|z| < R$ with boundary $\Gamma : |z| = R$ and let $U(\phi)$ be a continuous real function on the interval $[0, 2\pi]$ such that $U(0) = U(2\pi)$. Then the function $u(r, \theta)$ defined by the integral

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)U(\phi)}{R^2 + r^2 - 2Rr\cos(\phi - \theta)}\, d\phi \qquad (33)$$

for any point (r, θ) on D any by $u(R, \phi) = U(\phi)$ (34)

for any point (R, ϕ) on Γ, solves the Dirichlet problem for the disc D. In otherwords, (i) u is harmonic on D and continuous on $\overline{D}$ and (ii) $\lim_{re^{i\theta} \to Re^{i\phi_0}} u(r, \theta) = U(\phi_0)$,

where $Re^{i\phi_0}$ is any fixed point on Γ.

Proof : To prove that $u(r, \theta)$ defined by (33) on D is harmonic on D we observe that

$$\frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\phi - \theta)} = P(R, r, \phi - \theta)$$

$= \text{Re}\left(\frac{\varsigma + z}{\varsigma - z}\right)$, where P (R, r, ϕ–θ) is the Poisson Kernel and $\varsigma = Re^{i\phi}$, $z = re^{i\theta}$, $r < R$.

The r.h.s. is the real part of the function $\frac{\varsigma + z}{\varsigma - z}$ which is analytic in D. Hence the Poisson Kernel P(R, r, ϕ–θ) is harmonic in D. So, differentiation under the sign of integration is valid. Applying the Laplacian ∇^2 in (r, θ) to both sides of (33) we get,

$\nabla^2 u = \frac{1}{2\pi}\int_0^{2\pi} U(\phi).\nabla^2 P(R, r, \phi - \theta)\, d\phi = 0$ [Since P(R, r, ϕ–θ)

is harmonic in D $\Rightarrow \nabla^2 P(R, r, \phi - \theta) = 0$].

$\Rightarrow$ u is harmonic on D.

Next we prove that the function u(r, θ) defined by the integral (33) approaches $U(\phi_0)$ as the point (r, θ) in D tends to any fixed point (R, ϕ_0) on Γ.

Let (r_n, θ_n) be an arbitrary sequence of points in D converging to the boundary point (R, ϕ_0). We now consider the difference

$$u(r_n, \theta_n) - U(\phi_0) = \frac{1}{2\pi}\int_0^{2\pi} P(R, r_n, \phi - \theta_n) U(\phi) d\phi - U(\phi_0)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \{U(\phi) - U(\phi_0)\} P(R, r_n, \phi - \theta_n) d\phi \qquad (35)$$

$$\left(\text{Since } \frac{1}{2\pi}\int_0^{2\pi} P(R, r_n, \phi - \theta_n) d\phi = 1\right)$$

Since U(ϕ) is continuous on Γ, for given $\in > 0$ there exists a $\delta(\in) > 0$ such that

$$|U(\phi) - U(\phi_0)| < \frac{\in}{2} \qquad (36)$$

whenever $\quad |\phi - \phi_0| < 2\delta \qquad (37)$

we choose δ so small that (36) is satisfied and $\phi_0 - 2\delta > 0$, $\phi_0 + 2\delta < 2\pi$. We break the integral on r.h.s. of (35) as

$$|u(r_n, \theta_n) - U(\phi_0)| \le \left|\frac{1}{2\pi}\int_0^{\phi_0 - 2\delta} P(R, r_n, \phi - \theta_n)\{U(\phi) - U(\phi_0)\} d\phi\right|$$

$$+ \left|\frac{1}{2\pi}\int_{\phi_0 - 2\delta}^{\phi_0 + 2\delta} \cdots\right| + \left|\frac{1}{2\pi}\int_{\phi_0 + 2\delta}^{2\pi} \cdots\right| = |I_1| + |I_2| + |I_3| \qquad (38)$$

Now, $$|I_2| \leq \frac{1}{2\pi}\int_{\phi_0-2\delta}^{\phi_0+2\delta} |P(R, r_n, \phi - \theta_n)|U(\phi) - U(\phi_0)|\, d\phi$$

$$< \frac{\in}{2}\cdot\frac{1}{2\pi}\int_0^{2\pi} |P(R, r_n, \phi - \theta_n)|\, d\phi = \frac{\varepsilon}{2} \qquad (39)$$

To estimate the other two terms we choose n so large that

$|\phi_0 - \theta_n| < \delta$. Then, $|\phi - \theta_n| = |\phi - \phi_0 + \phi_0 - \theta_n| \geq |\ \phi - \phi_0| - |\phi_0 - \theta_n) > 2\delta - \delta = \delta$ since $|\phi - \phi_0| > 2\delta$ whenever ϕ belongs to either of the intervals $[0, \phi_0 - 2\delta]$ or, $[\phi_0 + 2\delta, 2\pi]$.

Then, $$|I_1| + |I_3| \leq 2M.\frac{1}{2\pi}.\frac{R^2 - r_n^2}{R^2 + r_n^2 - 2Rr_n \cos\delta}\left(\int_0^{\phi_0-2\delta} d\phi + \int_{\phi_0+2\delta}^{2\pi} d\phi\right)$$

$$< 2M\frac{R^2 - r_n^2}{R^2 + r_n^2 - 2Rr_n \cos\delta} \to 0 \text{ as } r_n \to R,$$

where $M = \underset{\phi \varepsilon [0, 2\pi]}{\text{Max}} |U(\phi) - U(\varphi_0)|$ and $\cos(\phi - \theta_n) < \cos\delta$.

Thus, for sufficiently large n, $|I_1| + |I_3| < \frac{\varepsilon}{2}$ (40)

Using (39) and (40) in (38) we get,

$|u(r_n, \theta_n) - U(\phi_0)| < \varepsilon$ for sufficiently large n;

i.e. $\lim_{n\to\infty} u(r_n, \theta_n) = U(\phi_0)$ (41)

where (r_n, θ_n) is an arbitrary sequence of points in D approaching (R, ϕ_0).

Equation (41) still holds if some or all the points (r_n, θ_n) lie on Γ since in that case we can directly use the fact that $U(\phi)$ is continuous on Γ. This implies $u(r, \theta)$ is continuous on $\overline{D}$. This completes the proof.

Uniqueness of the solution to the Dirichlet problem for a disc.

Let u_1 and u_2 be two solutions of the Dirichlet problem. Then their difference $u_1 - u_2 = h$ is harmonic in D and continuous in the closed disk and takes the value zero on the boundary. Hence h attains its upper bounds at some points of the closed disk. If l > 0, the upper bound will occur in the open disk, since on the boundary Γ h is zero. This contradicts the conclusions of theorem 2.2. So then $l = 0$. In the same way we can show that the lower bound of h on $\overline{D}$ is zero. Thus there is no alternative but h to be zero on $\overline{D}$.

Theorem 2.4 Any continuous function u(z) possessing the mean-value property in a domain D is harmonic in D.

Proof. Let $\overline{K}$ be a closed disk contained in D. By hypothesis of the theorem u satisfies the mean value property in K. We shall prove that u is harmonic in K. By the theorem 2.3 on the Dirichlet problem for a disk there exists a continuous function $\tilde{u}(z)$ in K, which is harmonic in the interior of K and coincides with u(z) on the boundary of K. The difference $u-\tilde{u}$ is continuous and satisfies the mean-value property in K. By the corollary to the theorem 3.7 [(14) page-58] $u-\tilde{u}$ satisfies the maximum modulus prnciple in K. Now as $u-\tilde{u}$ is zero on the boundary of K, it will be identically zero in K. Therefore u coincides with the harmonic function $\tilde{u}$ in the interior of K and since K is arbitrary, u is harmomic in the domain D.

The Harnack Inequality : Let u be a non-negative Harmonic function on a closed disk $\overline{D}$ (0, R). Then, for any point z ε D(0, R)

$$\frac{R-|z|}{R+|z|}u(0) \le u(z) \le \frac{R+|z|}{R-|z|}u(0) \tag{42}$$

where D(0, R) denotes a disk with centre 0 and radius R.

Proof. From the Poisson's integral formula for u on $\overline{D}$ (0, R) :

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\phi})\frac{R^2-|z|^2}{|Re^{i\phi}-z|^2}d\phi$$

Now, $$\frac{R^2-|z|^2}{|Re^{i\phi}-z|^2} \le \frac{R^2-|z|^2}{(R-|z|)^2} = \frac{R+|z|}{R-|z|}$$

Combining these two, we see that

$$u(z) \le \frac{R+|z|}{R-|z|}\frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\phi})\,d\phi = \frac{R+|z|}{R-|z|}u(0),$$

where we make use of the mean value theorem. Similarly, the other inequality in (42) will follow from $\frac{R^2-|z|^2}{|Re^{i\phi}-z|^2} \ge \frac{R^2-|z|^2}{(R+|z|)^2} = \frac{R-|z|}{R+|z|}$

Corollary Let u be a non-negative harmonic function on a closed disk $\overline{D}(\varsigma, R)$. Then for any z ε D (ς, R),

$$\frac{R-|z-\varsigma|}{R+|z-\varsigma|}u(\varsigma) \le u(z) \le \frac{R+|z-\varsigma|}{R-|z-\varsigma|}u(\varsigma) \tag{43}$$

2.5 Subharmonic & Superharmonic Functions

Definition : A real-valued continuous function u(x, y) in an open set D of the complex plane $\mathbb{C}$ is said to be

(i) subharmonic if, for any $\varsigma \in D$

$$u(\varsigma) \le \frac{1}{2\pi}\int_0^{2\pi} u(\varsigma + re^{i\theta})\,d\theta$$

hold for sufficiently small r > 0.

(ii) superharmonic if, for any $a \in D$

$$u(a) \ge \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})\,d\theta$$

hold for sufficiently small r > 0.

From the definition it follows that every harmonic function is subharmonic as well as superharmonic.

Example 3. If f(z) is analytic on a domain D, then |f(z)| is subharmonic but not harmonic in D unless f(z) ≡ constant.

Solution : Using the Cauchy's integral formula

$$|f(a)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(a + re^{i\theta})|\,d\theta \tag{44}$$

for every $a \in D$ and r (> 0) is small enough. Here equality holds only if f(z) ≡ constant. We now show that the integral

$$I(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(a + re^{i\theta})|\,d\theta$$

is a strictly increasing function of r, if f(z) ≠ constant. Let $0 < r_1 < r_2 < k(a)$ and g(θ) be continuous on [0, 2π] and F(z) be defined by

(i) $g(\theta)f(a + r_1e^{i\theta}) = |f(a + r_1e^{i\theta})|, 0 \le \theta \le 2\pi$

(ii) $F(z) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(a + ze^{i\theta})\,g(\theta)d\theta,\ |z| \le r_2$

(iii) k(a) ≡ minmum distance between a and the boundary of D.

F(z) is regular for $|z| \le r_2$ and attains its maximum of the boundary of the disc, say at $z = r_2e^{i\phi}$. Then

$$I(r_1) = \frac{1}{2\pi}\int_0^{2\pi} |f(a + r_1e^{i\theta})|\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(a + r_1e^{i\theta})\,g(\theta)d\theta$$

$= F(r_1)$

$< \left|F(r_2 e^{i\theta})\right|$

$\leq \frac{1}{2\pi}\int_0^{2\pi}\left|f(a + r_2 e^{i(\theta+\phi)})\right| d\theta$

$= \frac{1}{2\pi}\int_{\phi}^{2\pi+\phi}\left|f(a + r_2 e^{i\psi})\right| d\psi,$ taking $\phi + \theta = \psi$

$= \frac{1}{2\pi}\left\{\int_0^{2\pi} - \int_0^{\phi} + \int_{2\pi}^{2\pi+\phi}\left|f(a + r_2 e^{i\psi})\right| d\psi\right\}$

$= \frac{1}{2\pi}\int_0^{2\pi}\left|f(a + r_2 e^{i\psi})\right| d\psi,$ (substituting $\psi = 2\pi + \theta$ in the third integral, we find that it cancels the second term)

$= I(r_2)$. Hence equality in (44) is possible if and only if $f(z) \equiv$ constant. Therefore $|f(z)|$ is subharmonic but not harmonic in D unless $f(z) \equiv$ constant.

Example 4. If $f(z) \neq 0$ is analytic in a domain D, then $\log |f(z)|$ is subharmonic in D.

Solution : Let $\Phi(z) = \log|f(z)|$. Here at the zeros of f(z), $\Phi(z)$ has poles and takes the value $-\infty$ there. In every closed disk contained in D there are at most a finite number of points where $\log f(z) = -\infty$.

Now let $a \in D$ be any point at which f(z) is distinct from zero. Since f(z) is analytic and not identically zero, there exists a small neighbourhood of a where f(z) is distinct from zero. We find that

$$\log f(z) = \log |f(z)| + i \arg f(z)$$

is analytic in this neighbourhood and hence $\log |f(z)|$ is harmonic there and we have the equality

$$\Phi(a) = \frac{1}{2\pi}\int_0^{2\pi}\Phi(a + re^{i\theta})\, d\theta \qquad (45)$$

for all sufficiently small values of r. On the otherhand, if a is a zero of f(z), we have

$$\Phi(a) = -\infty < \frac{1}{2\pi}\int_0^{2\pi}\Phi(a + re^{i\theta})\, d\theta \qquad (46)$$

Combining (45) with (46) we obtain $\Phi(z)$ is subharmonic in D.

Unit 3 ❑ Conformal Mappings

Structure

3.0 Objectives of this Chapter

This chapter deals with conformal mappings and their basic properties. Many examples are given to explain different concepts on conformal mappings. The inverse function theorem is also discussed.

3.1 Conformal Mappings

Let X be an open set in $\mathbb{C}$ and suppose a function $f : X \to \mathbb{C}$ is given. We know from functional analysis that if f is continuous, a compact set of X is mapped onto a compact set in f(X) and a connected set of X onto a connected set of f(X). If moreover, f is single-valued and analytic there occur several interesting results. In this chapter we study mappings which transform different curves and regions from one complex plane to other complex plane with reference to magnitude and orientation. Such type of mappings play an important role in the study of various physical problems defined on domains and curves of arbitrary shape.

Level Curves

Let $w = f(z)$ with $z = x + iy$ and $w = u + iv$ where f(z) is analytic. $u = u(x, y)$ $v = v(x, y)$ satisfy Cauchy-Riemann equations

$$u_x = v_y,\ u_y = -v_x$$

from which it follows that

$$u_{xx} + u_{yy} = 0$$

$$v_{xx} = v_{yy} = 0$$

Also, $\nabla_u . \nabla_v = 0$, where

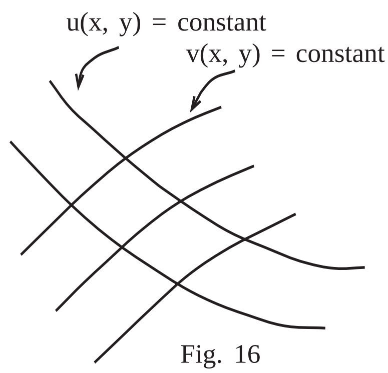


Fig. 16

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

So that the level curves u (x, y) = constant and v (x, y) = constant are orthogonal.

Now $f^1(z) = u_x + iv_x = u_x - iu_y = v_y + iv_x$

so that $|f^1(z)|^2 = u_x^2 + u_y^2 = v_x^2 + v_y^2.$

Two basic results :

No. 1

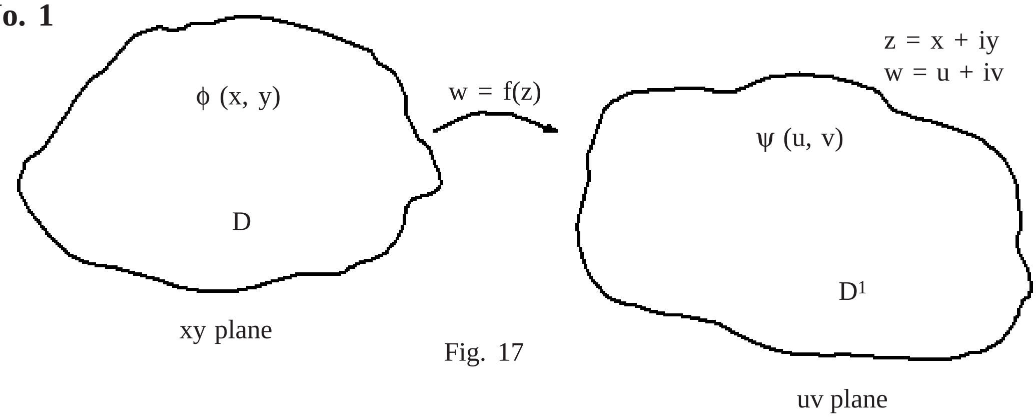


Fig. 17

Suppose that w = f(z) maps D into D^1.

Let $\psi(u, v) = \psi((u(x, y), v(x, y)) = \phi(x, y)$.

To prove $\phi_{xx} + \phi_{yy} = |f^1(z)|^2 (\psi_{uu} + \psi_{vv})$

we calculate $\phi_x = \psi_u u_x + \psi_v v_x$

$$\phi_{xx} = \psi_{uu} u_x^2 + \psi_{vv} v_x^2 + 2\psi_{uv} u_x v_x + \psi_u u_{xx} + \psi_v v_{xx}$$

$$\phi_{yy} = \psi_{uu} u_y^2 + \psi_{vv} v_y^2 + 2\psi_{uv} u_y v_y + \psi_u u_{yy} + \psi_v v_{yy}$$

Thus, $\phi_{xx} + \phi_{yy} = (u_x^2 + u_y^2)\psi_{uu} + (v_x^2 + v_y^2)\psi_{vv} + 2\psi_{uv}\nabla_u . \nabla_v,$

since u, v satisfy Laplace equation. Again, $\nabla_u . \nabla_v = 0$,

so we obtain $\phi_{xx} + \phi_{yy} = |f^1(z)|^2 (\psi_{uu} + \psi_{vv})$

Therefore if $f^1(z) \neq 0$ inside D we have $\phi_{xx} + \phi_{yy} = 0$ imples $\psi_{uu} + \psi_{vv} = 0$ and vice-versa.

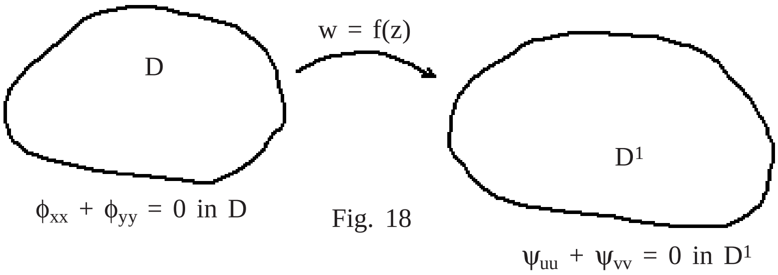


Fig. 18

No. 2. Consider a level curve F(x, y) = 0 upon $\nabla\phi \,.\, \underline{n} = 0$.

Let under the analytic mapping w = f(z) the level curve map to G(u, v) = 0.

We shall show that $\nabla\psi.\underline{n} = 0$ on G(u, v) = 0

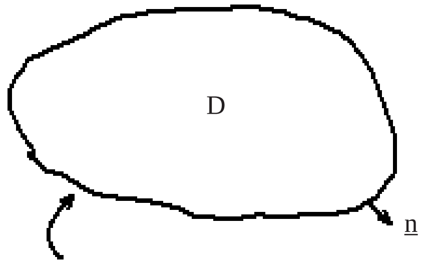


Fig. 19

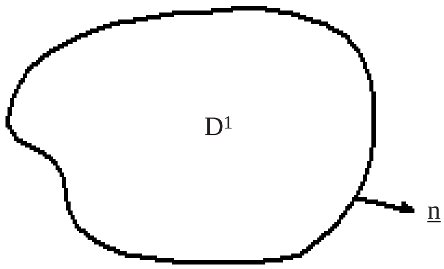


Fig. 20

Consider the map w = f(z) $\to$ ω = u + iv, so u = u(x, y), v = v(x, y).

Suppose f(z) is analytic. Then,

$$\left.\begin{aligned}\phi_x &= \psi_u u_x + \psi_v v_x \\ \phi_y &= \psi_u u_y + \psi_v v_y\end{aligned}\right\} \quad \text{so,} \quad \begin{pmatrix}\phi_x \\ \phi_y\end{pmatrix} = S\begin{pmatrix}\psi_u \\ \psi_v\end{pmatrix} \text{ with } S = \begin{pmatrix}u_x v_x \\ u_y v_y\end{pmatrix}$$

Then, $\nabla\phi = S\nabla_\psi$, $\nabla F = S\nabla G$ and clearly, $S^T S = |f^1(z)|^2\, 1$

Now, $$\frac{\partial\phi}{\partial\underline{n}} = \nabla\phi\cdot\frac{\nabla F}{|\nabla F|} = \frac{S\nabla\psi.(S\nabla G)}{|S\nabla G|} = \frac{(\nabla\psi)^T S^T S\nabla G}{\left\{(S\nabla G)^T(S\nabla G)\right\}^{1/2}} = \frac{(\nabla\psi)^T\nabla G\left|f^1(z)\right|}{\left\{(\nabla G)^T\nabla G\right\}^{1/2}}$$

(where the usual vector operations, $\underline{a}.\underline{b} = a^T b$ and $(a.a)^{1/2} = (a^T a)^{1/2} = |a|$ have been used)

So, $$\frac{\partial\phi}{\partial\underline{n}} = \nabla\phi\cdot\frac{\nabla F}{|\nabla F|} = \left|f^1(z)\right|\nabla\psi\frac{\nabla G}{|\nabla G|} = \left|f^1(z)\right|\frac{\partial\psi}{\partial\underline{n}}$$

This shows that if $\frac{\partial\phi}{\partial\underline{n}} = 0$ on the boundary of D then $\frac{\partial\psi}{\partial\underline{n}} = 0$ on the boundary of D^1, provided $|f^1(z)| \neq 0$ on the boundary of D.

Note : These give us a means of transforming the domain over which the Laplace's equation is to be solved comfortably. Such type of things is usually dealt in solving boundary value problems in potential theory.

Angle of Rotation

Given a function of a complex variable w = f(z) analytic in a domain D. Let z_0 be any point lying within D, $\gamma : z = \sigma(t)$, $a \le t \le b$, $\sigma(t_0) = z_0$, be a curve passing through z_0 (and lying within D). The function $\sigma(t)$ has a non zero derivative $\sigma^1(t_0)$ at the point z_0 and the curve γ has a tangent at this point with a slope equal to Arg $\sigma^1(t_0)$.

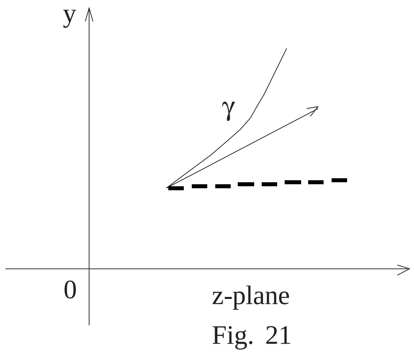


Fig. 21

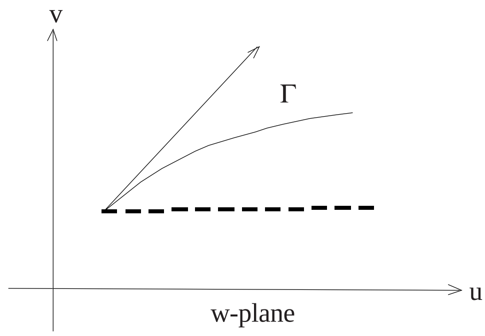


Fig. 22

Under the mapping $w = f(z)$ the curve γ is transformed into a curve $\Gamma : w = f(\sigma(t)) = \mu(t)$, $a \le t \le b$, $\mu(t_0) = f(z_0) = w_0$ in the w-plane. $\mu(t)$ is differentiable at $t = t_0$ and the curve Γ has a tangent at $w_0 = f(z_0)$. Then following the chain rule for differentiation of composite functions, assuming $f^1(z_0) \neq 0$

$$\mu^1(t_0) = f^1(\sigma(t_0)\ \sigma^1(t_0)$$

It follows that

$$\text{Arg } \mu^1(t_0) = \text{Arg } f^1(z_0) + \text{Arg } \sigma^1(t_0)$$

i.e.,
$$\text{Arg } \mu^1(t_0) = \text{Arg } \sigma^1(t_0) + \text{Arg } f^1(z_0) \tag{47}$$

This implies that change in slope of a curve at a point under a transformation depends only on the point and not on the particular curve through that point.

Example 1. Verify the result given in equation (47) for the curve $y = x^2$ under the transformation $f(z) = z^2$ at $z = 1 + i$.

Solution. First we calculate the change in slope of the curve $y = x^2$ at the given point under the transformation $w \equiv f(z) = z^2$. Following the formula given in eq. (47)

$$\text{Arg } f^1(1 + i) = \text{Arg } 2(1 + i) = \tan^{-1} 1$$

A parametric form of the given curve $y = x^2$ is given by

$$\gamma : z = t + it^2, -\infty < t < \infty.$$

Here $z_0 = 1 + i$ at $t_0 = 1$ and $z^1(1) = 1 + 2i$, so that slope of the curve γ is $\tan^{-1} 2$.

Now we find slope of the transformed curve.

$$w = f(z) \Rightarrow u + iv = (x + iy)^2$$

So, $u = x^2 - y^2$ and $v = 2xy = 2x \,.\, x^2 = 2x^3$.

Then, $u = x^2 - x^4 = \left(\frac{v}{2}\right)^{2/3} - \left(\frac{v}{2}\right)^{4/3}$, which is the equation of the transformed curve Γ. The image of the point $(1 + i)$ of z-plane is the point $2i$ in the w-plane and the slope of the curve Γ at $w = 2i$ is

$$\left.\frac{dv}{du}\right|_{w=2i} = -3$$

Thus the change in slope of the curve γ under the transformation is

$$\tan^{-1}(-3) - \tan^{-1}(2) = \tan^{-1}\frac{-3-2}{1-6} = \tan^{-1} 1$$

which is the same as obtained earlier following equation (47).

Definition : A mapping $w = f(z)$ is said to be conformal at a point $z = z_0$, if it preserves angles between oriented curves, passing through z_0, in magnitude and in sense of rotation.

Theorem 3.1 : Let $f(z)$ be an analytic function in a domain D containing z_0. If $f^1(z_0) \neq 0$, then $f(z)$ is conformal at z_0.

Proof. Let $C_1 : z = z_1(t)$ and $C_2 : z = z_2(t)$, $t \equiv$ parameter, be two curves which intersect at some $t = t_0$ where $z_1(t_0) = z_2(t_0) = z_0$, C_1^1, C_2^1 are their images under the mapping $w = f(z)$.

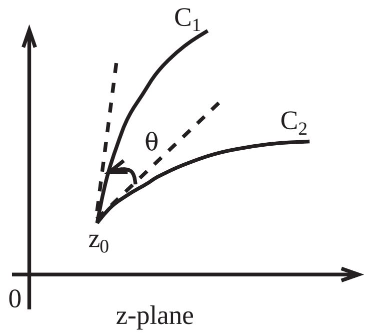


Fig. 21

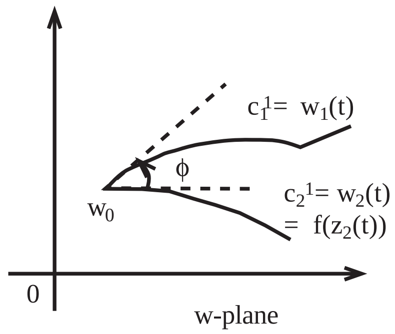


Fig. 22

tangent lines are

$z^1 = z_1{}^1(t_0)$, $z^1 = z_2{}^1(t_0)$ at $t = t_0$

tangent lines are

$w_1{}^1(t_0) = f^1(z_1(t_0)z_1{}^1(t_0))$

$w_2{}^1(t_0) = f^1(z_2(t_0)z_2{}^{(t_0)}z_2{}^1(t_0))$

Then following the result given in eq. (47)

$$\text{Arg}(w_1^1(t_0)) - \text{Arg}(z_1^1(t_0)) = \text{Arg}(f^1(z_1(t_0)) = \text{Arg}(f^1 z_0))$$

and

$$\text{Arg}(w_2^1(t_0)) - \text{Arg}(z_2^1(t_0)) = \text{Arg}(f^1(z_2(t_0)) = \text{Arg}(f^1 z_0)).$$

Subtracting, $\text{Arg}(w_1^1(t_0)) - \text{Arg}(w_2^1(t_0)) - \left\{\text{Arg}(z_1^1(t_0)) - \text{Arg}(z_2^1(t_0))\right\} = 0$

i.e., $\theta = \phi$, where θ = angle between the curves C_1 and C_2 at z_0 and
ϕ = angle between the curves C_1^1 and C_2^1 at w_0.

Observation : From the basic results proved earlier we learn that if f is a conformal mapping, then orthogonal curves are mapped onto orthogonal curves.

3.2 Basic Properties of conformal Mappings

Let f(z) be an analytic function in a domain D, and let z_0 be a point in D. If $f(z_0) = 0$, then we can express f(z) in the form

$$f(z) = f(z_0) + (z - z_0)f^1(z_0) + (z - z_0)\eta(z),$$

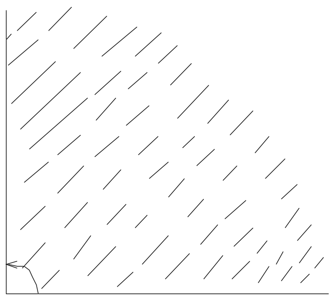
where $\eta(z) \to 0$ as $z \to z_0$. If z is near z_0, then the transformation w = f(z) has the linear approximation

$$G(z) = A + B(z - z_0).$$

where $A = f(z_0)$ and $B = f^1(z_0)$. As $\eta(z) \to 0$ when $z \to z_0$, for points near z_n the transformation w = f(z) has an effect much like the linear mapping w = G(z). The effect of the linear mapping G is a rotation of the plane through the angle $\alpha = \text{Arg}(f^1(z_0))$, followed by a magnification by the factor $|f(z_0)|$, followed by a translation by the vector $A + BZ_0$.

Remark : If $f^1(z_0) = 0$, the angle may not be preserved.

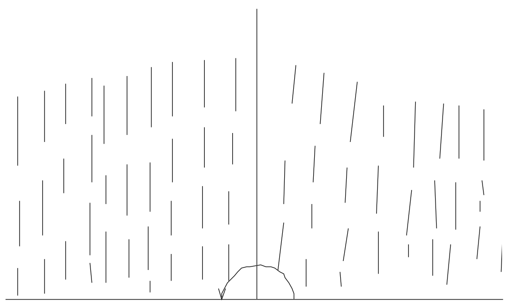
Let us consider, $w = f(z) = z^2$, then we have $f^1(0) = 0$ and



0

z-plane

Fig. 23



0

w-plane

Fig. 24

the angle at z = 0 is not preserved but is doubled.

Definition : Let f(z) be a nonconstant analytic function. If $f^1(z_0) = 0$, the z_0 is called a critical point of f(z), and the mapping w = f(z) is not conformal at z_0. We shall see afterwards what happens at a critical point.

The Inverse Function theorem 3.2 Let f(z) be analytic at z_0 and $f^1(z_0) \neq 0$. Then there exists a neighbourhood $N(w_0, \varepsilon)$ of $w_0 = f(z_0)$ in which the inverse function $z = F(w)$ exists and is analytic.

Moreover, $F^1(w_0) = 1/f^1(z_0)$. (48)

Proof : Given $w = f(z)$, $(z = x + iy,\ w = u + iv)$

is analytic in a neighbourhood of z_0, $K : |z - z_0| < \rho$. We shall show that for each $w \in L : |w - w_0| < \in$ there is a unique solution $z = F(w)$, where $z \in K$.

We express the mapping $w = f(z)$ in terms of the set of equations

$$u = u(x, y) \text{ and } v = v(x, y) \tag{49}$$

which represents a transformation from the xy plane to the uv plane, u, v, possess continuous first-order partial derivatives satisfying C-R equations. The Jacobian determinant J(x, y), is defined by

$$J(x,y) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \tag{50}$$

The transformation in equations (49) has a local inverse in L provided $J(x, y) \neq 0$ in K [(3) pp. 358-361]. Expanding r.h.s. of equation (50) and using the C-R equations, we obtain

$$\begin{aligned} J(x_0, y_0) &= u_x^2(x_0, y_0) + v_x^2(x_0, y_0) \\ &= |f^1(z_0)|^2 \qquad (51) \\ &\neq 0, \text{ by the given hypothesis.} \end{aligned}$$

Utilising the continuity of J(x, y) in a small neighbourhood of (x_0, y_0), equations (49) and (51) imply that a local inverse $z = F(w)$ exists in a neighbourhood of the point $w_0 = f(z_0)$. The derivative of F(w) is given by the familiar expression

$$F^1(w) = \lim_{\Delta w \to 0} \frac{F(w + \Delta w) - F(w)}{\Delta w} = \lim_{\Delta w \to 0} \frac{\Delta z}{\Delta w} = \lim_{\Delta z \to 0} \frac{\Delta z}{f(z + \Delta z) - f(z)}$$

$$= \lim_{\Delta z \to 0} 1 \Big/ \left(\frac{f(z + \Delta z) - f(z)}{\Delta z} \right) = 1 \Big/ \left(\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \right)$$

i.e., $F^1(w) = \dfrac{1}{f^1(z)}$

holds in a neighbourhood of the point w_0, as f(z) is analytic in K.

In particular, $F^1(w_0) = \dfrac{1}{f^1(z_0)}$

Theorem 3.3 Let f(z) be analytic at the point z_0. If $f^1(z_0) = 0$, $f^{11}(z_0) = 0$, ...,

$f^{(k-1)}(z_0) = 0$ and $f^{(k)}(z_0) \neq 0$, then the mapping $w = f(z)$ magnifies angles at z_0 by k times.

Proof. By the given hypothesis, f(z) has the Taylor expansion in a neighbourhood of z_0 in the form

$$f(z) = f(z_0) + c_k(z - z_0)^k + c_{k+1}(z - z_0)^{k+1} + \ldots, \; c_k \neq 0$$

so that we can express

$$f(z) - f(z_0) = (z - z_0)^k + h(z) \qquad (52)$$

where h(z) is analytic at z_0 and $h(z_0) \neq 0$. Now let $w = f(z)$ and $w_0 = f(z_0)$ and we obtain from (52)

$$\text{Arg}(w - w_0) = k \; \text{Arg}(z - z_0) + \text{Arg}(h(z))$$

Let $z \to z_0$ along a curve γ. Then $w \to w_0$ along the image curve Γ and the slope of tangent to the curve γ at z_0 and that of the tangent to the curve Γ at w_0 are connected by the relation

$$\lim_{w \to w_0} \text{Arg}(w - w_0) = k \lim_{z \to z_0} \text{Arg}(z - z_0) + \lim_{z \to z_0} \text{Arg}(h(z))$$

i.e., $\qquad \theta_0 = k\phi_0 + \text{Arg}(h(z))$

Thus, if γ_1 and γ_2 be two curves passing through z_0 and their images Γ_1 and Γ_2 under the mapping $w = f(z)$, pass through w_0, the difference of slopes of the curves γ_1 and γ_2 at z_0 and that of the curves Γ_1 and Γ_2 at w_0 are related as

$$\theta_2 - \theta_1 = k(\phi_2 - \phi_1)$$

with the sense remain unchanged.

Example 2. Show that the mapping $w = f(z) = z^2$ maps the rectangle

$R = \left\{ x + iy : -1 \leq x \leq 1, \; 0 \leq y \leq \frac{1}{2} \right\}$ of unit area onto the region enclosed by the parabolas

$$v^2 = u + \frac{1}{4} \text{ and } v^2 = -4(u - 1).$$

Solution : Here $f^1(z) = 2z$ and the mapping $w = z^2$ is conformal for all $z \neq 0$. We note that the right angles at the vertices $z_1 = 1$, $z_2 = 1 + i/2$, $z_3 = -1 + i/2$ and $z_4 = -1$ are mapped into right angles at the vertices $w_1 = 1, \; w_2 = \frac{3}{4} + i, \; w_3 = \frac{3}{4} - i$ and $w_4 = 1$ respectively.

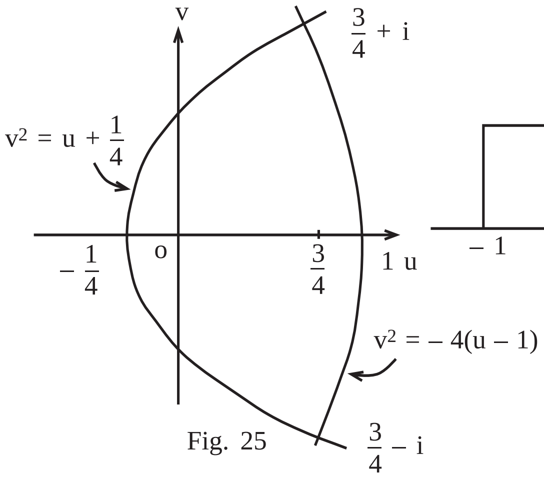


Fig. 25

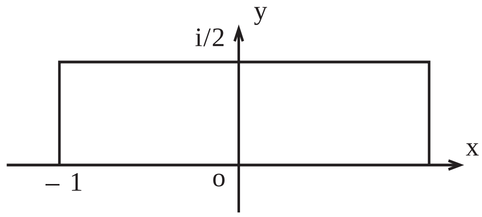


Fig. 26

The parabolas shown in the figure are obtained as follows :

Let $w = u + iv$. Then $u = x^2 - y^2,\ v = 2xy$ }... (53)

The line $x = 1$ corresponds to the curve $u = 1 - y^2,\ v = 2y$. Eliminating y, we get $v^2 = -4(u-1)$, which is a parabola with vertex (1, 0) and opens towards the negative side of the u-axis in the w-plane. Also, the part of the line $x = 1$ lying above the real axis corresponds to the part of the parabola lying above the u-axis in the w-plane. The same parabola in the w-plane is the image of the line $x = -1$. In this case, the part of the line $x = -1$ lying above the real axis corresponds to the part of the parabola lying below the u-axis in the w-plane.

Again, when $y = \frac{1}{2}$, from (53) we get $u = x^2 - \frac{1}{4}$ and $v = x$. Eliminating x we get, $v^2 = u + \frac{1}{4}$ which is also a parabola with vertex $\left(-\frac{1}{4},\ 0\right)$ and opening towards the positive side of the u-axis in the w-plane. By similar argument as before we can say that the mapping $w = z^2$ maps the rectangle $R = \left\{x + iy : -1 \le x \le 1,\ 0 \le y \le \frac{1}{2}\right\}$ onto the region enclosed by the parabolas $v^2 = u + \frac{1}{4}$ and $v^2 = -4(u-1)$.

Note : It is not hard to prove that the parabolas intersect each other orthogonally at w_2 and w_3.

At the point $z_0 = 0$, we have $f^1(z_0) = f^1(0) = 0$ and $f^{11}(z_0) = 2 \neq 0$. Hence the angles at the origin $z_0 = 0$ are magnified by the factor $k = 2$. In particular the straight angle at $z_0 = 0$ is mapped onto 2π angle at $w_0 = 0$.

Unit 4 ❑ Multi-valued functions and Riemann Surface

Structure

4.0 Objectives of this Chapter

In this chapter we shall study multi-valued functions and their Riemann surfaces. In particular, multi-valued logarithm function, the power function z^{α} both z, α complex numbers, $z \neq 0$ will be discussed. The ideas of branch, branch point, branch cut, branch point at infinity will be explained by means of different examples. A few contour integrations of multi-valued functions will be performed. Also Riemann surfaces for different multi-valued functions will be constructed.

4.1 Multi-valued functions

So far we have considered single-valued functions i.e., one-to-one mapping or, many-to-one mapping. In the later case, under certain restrictions, inverse mappings give rise to multi-valued functions i.e., one-to-many.

For example,

$$z = e^{\omega},\ z = \omega^2,\ z = \sin\omega,\ z = \cos\omega$$

For each of these functions, a given value of z corresponds to more than one value of ω.

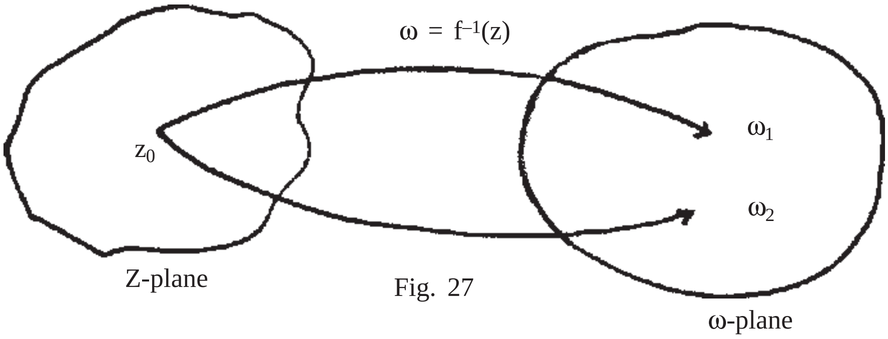


Fig. 27

$\omega = f^{-1}(z)$ is multi-valued and $z = f(\omega)$ is single-valued, given ω, there is a unique value of z.

The aim of this chapter is as follows :

(i) To determine all possible values of the inverse function ω and (ii) To construct an inverse function which is single-valued in some region of the complex plane.

Let $\omega = f(z)$ be a multi-valued function. A branch of f is any single-valued function f_0 that is continuous in some domain (except, perhaps, on the boundary). At each point z in the domain, it assigns one of the values of f(z).

Example 1 : We consider branches of the two-valued square-root function $f(z) = z^{1/2}(z \neq 0)$. The principal branch of the square root function is

$$f_1(z) = |z|^{1/2} e^{i\theta/2} = r^{1/2}\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right), \theta = \operatorname{Arg}(z)$$

where $r = |z|$ and $-\pi < \theta \leq \pi$. The function f_1 is a branch of f. Using the same notation, we can find other branches of the function f. For example if we let

$$f_2(z) = |z|^{1/2} e^{i(\theta+2\pi)/2} = r^{1/2}\left[\cos\left(\frac{\theta+2\pi}{2}\right) + i\sin\left(\frac{\theta+2\pi}{2}\right)\right]$$

then

$$f_2(z) = r^{1/2} e^{i(\theta+2\pi)/2} = r^{1/2} e^{i\theta/2}.e^{i\pi} = -f_1(z).$$

So, f_1 and f_2 can be taken as the two branches of the multi-valued square root function. The negative real axis is called a branch cut for the functions f_1 and f_2. Each point on the branch cut is a point of discontinuity for both functions f_1 and f_2.

Result 1 : Show that the function f_1 is discontinuous on the negative real axis.

Solution : Let $z_0 = r_0e^{i\pi}$ be any point on the negative real axis. We compute the limit as z approaches z_0 through the upper half plane lm z > 0 and the limit as z approaches z_0 through the lower half plane lm z < 0. The limits are

$$\lim_{(r,\theta)\to(r_0,\pi)} f_1(re^{i\theta}) = \lim_{(r,\theta)\to(r_0,\pi)} r^{1/2}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right] = ir_0^{1/2}, \text{ and}$$

$$\lim_{(r,\theta)\to(r_0,-\pi)} f_1(re^{i\theta}) = \lim_{(r,\theta)\to(r_0,-\pi)} r^{1/2}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right] = -ir_0^{1/2}$$

The two limits are distinct, so the function f_1 is discontinuous at z_0. Since z_0 is an arbitrary point on the negative real axis, f_1 is discontinous there.

Note : Likewise, f_2 is discontinuous at z_0.

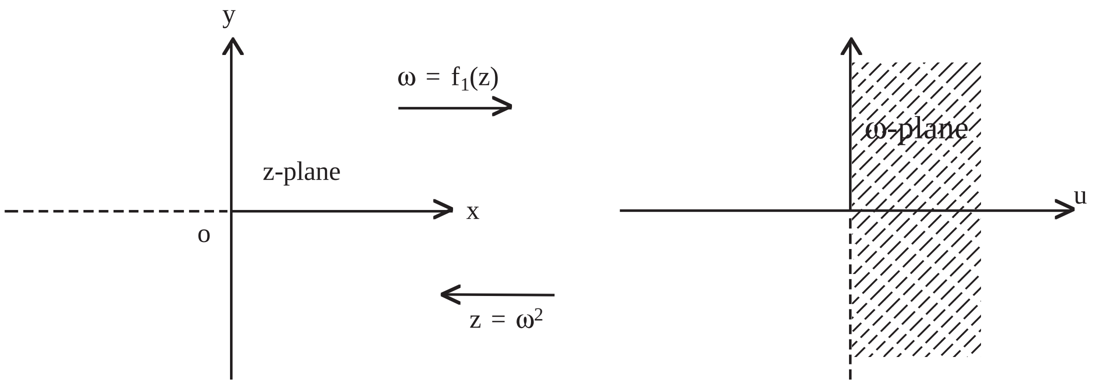


Fig. 28 a   Fig. 28 b

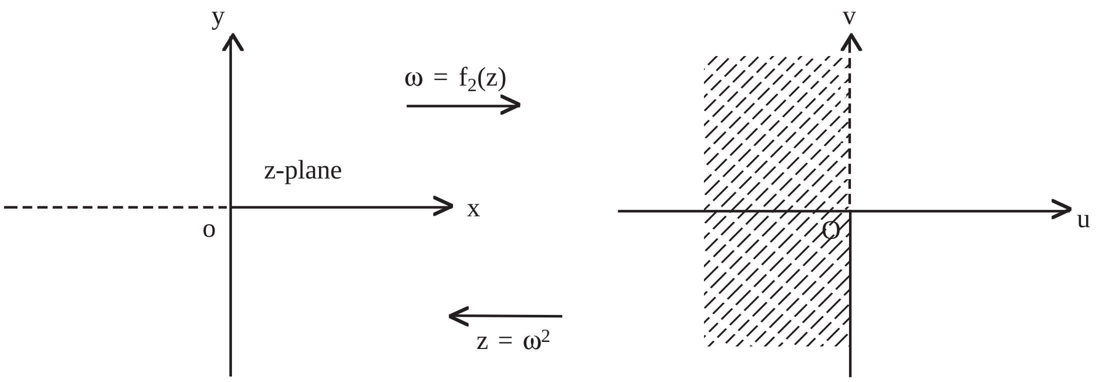


Fig. 29 a   Fig. 29 b

Figures : 28-29 The Branches f_1 and f_2 of $f(z) = z^{1/2}$

4.2 The logarithm function

Let us define the inverse function $f^{-1}(z)$ for $z = e^{\omega}$: Let $z = re^{i\theta}$ and $\omega = u + iv$. Then

$$re^{i\theta} = e^{u}.e^{iv}$$

So that $r = e^{u}$ and $v = \theta + 2k\pi,\ k = 0, \pm 1, \pm 2,...$

and $\omega = \log r + i(\theta + 2k\pi),\ k = 0, \pm 1, \pm 2,...$

But $r = |z|$ and without loss of generality, we can take $\theta \in (-\pi, \pi)$. This motivates the definition of the inverse function $f^{-1}(z)$ for $z = e^{\omega}$

$$\omega = \log z = \log|z| + i(\text{Arg } z + 2k\pi),\ k = 0, \pm 1, \pm 2,...$$

or, equivalently

$$\omega = \log z = \log|z| + i \arg z.$$

Mapping of the strip $|\text{Im } \omega| < \pi$ under $z = e^{\omega}$

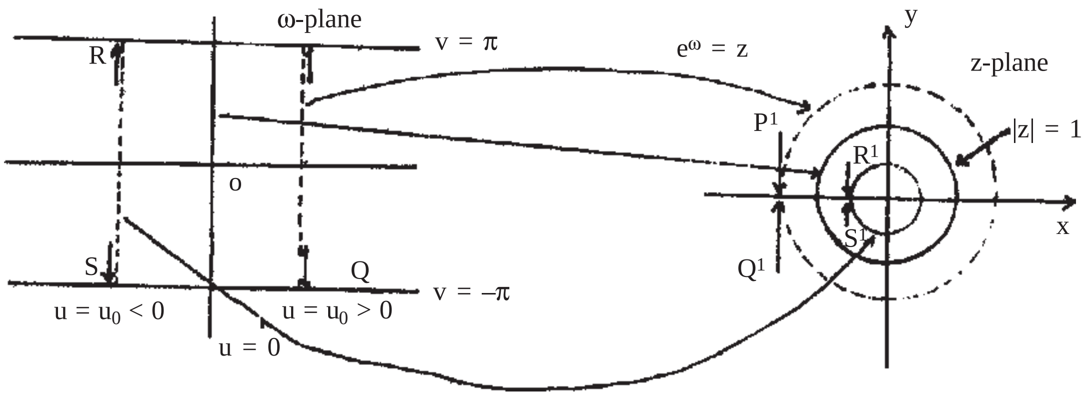


Fig. 30

I. Take $u = u_0 > 0,\ v \in (-\pi, \pi)$ for the line PQ :

$$x + iy = e_0^u(\cos v + i \sin v)$$

$$\Rightarrow \left.\begin{matrix} x = e^{u_0}\cos v \\ y = e^{u_0}\sin v \end{matrix}\right\} \rightarrow x^2 + y^2 = e^{2u_0} > 1,$$

a full circle in z-plane outside $|z| = 1$.

Now approach Q; $u = u_0 > 0,\ v = -\pi + \varepsilon$

$x = e^{u_0}\cos(-\pi + \varepsilon) \rightarrow -e^{u_0}$ as $\varepsilon \rightarrow 0+$ and $-e^{u_0} < -1$ as $u_0 > 0$

$y = e^{u_0}\sin(-\pi + \varepsilon) \rightarrow 0-$ as $\varepsilon \rightarrow 0+$

Now approach P : $u = u_0 > 0,\ v = \pi - \varepsilon$

$$x = e^{u_0}\cos(\pi - \varepsilon) \to -e^{u_0} \text{ as } \varepsilon \to 0+$$
$$y = e^{u_0}\sin(\pi - \varepsilon) \to 0+ \text{ as } \varepsilon \to 0+$$

II. Now take $u = u_0 < 0$, $v \in (-\pi, \pi)$ for the line RS :

$$\Rightarrow \quad \left.\begin{array}{l} x = e^{-u_0}\cos v \\ y = e^{-u_0}\sin v \end{array}\right\} \to x^2 + y^2 = e^{-2u_0} < 1$$

represents a full circle in z-plane inside $|z| < 1$.

Approach $\quad S : u = -u_0 < 0,\ v = -\pi + \varepsilon$

$$x = e^{-u_0}\cos(-\pi + \varepsilon) \to -e^{-u_0} > -1 \text{ as } \varepsilon \to 0+$$
$$y = e^{-u_0}\sin(-\pi + \varepsilon) \to 0- \text{ as } \varepsilon \to 0+$$

Now approach $\quad R : u = -u_0 < 0,\ v = \pi - \varepsilon$

$$x = e^{-u_0}\cos(\pi - \varepsilon) \to -e^{-u_0} > -1 \text{ as } \varepsilon \to 0+$$
$$y = e^{-u_0}\sin(\pi - \varepsilon) \to 0 \text{ as } \varepsilon \to 0+$$

Observation : Points along the negative real axis in the z-plane yield multiple w values. In order to obtain a single-valued inverse function for the fundamental strip $|\text{lm } \omega| < \pi$ we require a cut in z-plane along $\text{Re } z < 0$. The mapping $z = e^w$ and $w = f^{-1}(z)$ will be single-valued in $|\text{lm } w| < \pi$ and $z \in \mathbb{C} \setminus (\infty, 0)$.

Clearly the inverse function $w = \text{Log } z = \log |z| + i\text{Arg } z,\ -\pi < \text{Arg } z \le \pi$ is single-valued. We call this function the principal value of log z.

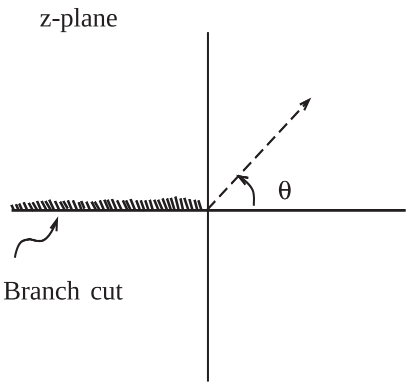


Fig. 31

The principal value of log z is not defined at z = 0 and is discontinuous as z approach the negative real axis from top and bottom. Using the necessary and sufficient conditions for differentiability we find

$$\frac{d}{dz}\text{Log } z = \frac{1}{z},\ z \ne 0,\ z \notin (-\infty, 0)$$

The point z = 0 is called a branch point of Log z since if we encircle the origin z = 0 by a closed contour then Log z changes by an amount proportional to $2\pi i$.

4.3 Properties of log z

(i) $\log (z_1 z_2) = \log z_1 + \log z_2$

(means that the set of all values of $\log z_1 + \log z_2$ is the same as the set of all values of $\log (z_1 z_2)$).

(ii) $z = e^{\log z}$, but $\log(e^z) = z + 2k\pi i$, $k = 0 \pm 1, \pm 2, \ldots$

Let $z = x + iy$

$$\log e^{x+iy} = \log(e^x) + i\left(\tan^{-1}\left(\frac{\sin y}{\cos y}\right) + 2k\pi\right) + x + iy = 2k\pi i$$

$$= z + 2k\pi i, \; k = 0, \pm 1, \ldots$$

(iii) $\log z^n \neq n \log z$ in general.

Let $z = re^{i\theta}$

$\log z^n = n \log r + i(n\theta + 2k\pi)$, $k = 0, \pm 1, \ldots$

$n \log z = n \log r + in(\theta + 2m\pi)$, $m = 0, \pm 1, \ldots$

Let n be fixed. Then the set of values of $\{k\}$, $k = 0, \pm 1, \pm 2, \ldots$

do not coincide with the set of values of $\{mn\}$, $m = 0, \pm 1, \pm 2, \ldots$

$$\Rightarrow \log z^n \neq n \log z$$

(iv) $\log\left(z^{1/n}\right) = \dfrac{1}{n}\log z$ (provided the set of values are the same) $n \equiv +$ ve integer.

Now, $z = re^{i\theta}$, $z^{1/n} = r^{1/n}e^{i(\theta + 2k\pi)/n}$, $k = 0, 1, 2, \ldots, n - 1$

$$\log z^{1/n} = \frac{1}{n}\log r + i\left(\frac{\theta + 2k\pi}{n} + 2\ell\pi\right), \; k = 0, 1, \ldots, n-1; \ell = 0, \; \pm 1, \; \pm 2, \ldots$$

Again, $$\frac{1}{n}\log z = \frac{1}{n}\log r + i\left(\frac{\theta}{n} + \frac{2m\pi}{n}\right), \; m = 0, \; \pm 1, \; \pm 2, \ldots$$

The set of values of $\log(z^{1/n})$ and $1/n \log z$ are the same if the sets $\{k + ln\}$, $k = 0, 1, \ldots, n - 1$; $l = 0, \pm 1, \pm 2, \ldots$ coincide with the set $\{m\}$, $m = 0, \pm 1, \pm 2, \ldots$

Complex exponents

If α is complex and $z \neq 0$ then

$z^\alpha = e^{\alpha \log z}$ multi-valued.

$z^\alpha = e^{\alpha[\log|z| + i(\operatorname{Arg} z + 2k\pi)]}$, $k = 0, \pm 1, \pm 2, \ldots$

$\quad = e^{\alpha[\log|z| + i(\theta + 2k\pi)]}$

agrees with our previous results if $\alpha = m$, $\alpha = \dfrac{1}{m}$; $m =$ integer. If α is a rational number say p/q, then z^α will have only q number of distinct values, occurred against $k = 0, 1, 2, \ldots, q - 1$ and the values of $e^{i2pk\pi/q}$ for $k = -1, -2, \ldots, -(q - 1)$ coincide with its values for $k = q - 1, q - 2, ..., 2, 1$ respectively, whereas the values of $e^{i2pk\pi/q}$ for $k = \pm q, \pm(q + 1), ...$ coincide with its values for $k = 0, \pm 1, \pm 2, ...$

z^{α} takes infinite number of values when α is irrational or complex. Clearly there is a distinct branch of z^{α} for each distinct branch of log z and the branch cuts are determined as in the case of log z. Every branch of z^{α} is analytic except at the branch point $z = 0$ and on a branch cut.

Example 2. Find all distinct values of i^{-2i}.

Solution : $i^{-2i} = e^{-2i \log i} = e^{2i\left[\log|i| + i\left(\frac{\pi}{2} + 2k\pi\right)\right]}, k = 0, \pm 1, \ldots$

$= e^{(4k+1)\pi}, \; k = 0, \pm 1, \pm 2, ...$

So, there are infinite number of values.

Example 3. Find all solutions of $z^{1-i} = 6$.

Solution : $e^{(1-i)\log z} = e^{\log 6}$

$\Rightarrow$ $(1 - i) \log z = \log 6 + 2k\pi i, \; k = 0, \pm 1, \pm 2, ...$

or, $2 \log z = (1 + i)[\log 6 + 2k\pi i]$

or, $$\log z = \frac{\log 6 - 2k\pi}{2} + \frac{i}{2}(\log 6 + 2k\pi)$$

Thus, $$z = e^{\log\sqrt{6} - k\pi}\left[\cos(k\pi + \log\sqrt{6}) + i\sin(k\pi + \log\sqrt{6})\right]$$

$$= \sqrt{6}e^{-k\pi}(-1)^k\left[\cos(\log\sqrt{6}) - i\sin(\log\sqrt{6})\right]$$

4.4 Branch, Branch point and Branch cut

Definition : F(z) is a **Branch** of the multi-valued function f(z) in a domain D if F(z) is single-valued and continuous in D and has the property that for each z in D the value of F(z) is one of the values of f(z).

To determine F(z) we introduce a line imanating from a point (called a **Branch Point**) to ensure that F is single-valued in the cut plane by the line. A **Branch Point** is one for which if we enclose it with a curve the function changes discontinuously as the variable makes a complete round over the curve.

For instance, consider $w = z^{1/2}$. Let P be a point on the z-plane where $w_1 = z_1^{1/2}$ and Arg $z_1 = \phi_1$, $0 < \phi_1 < 2\pi$.

Let $z_1 = r_1 e^{i\phi_1}$, then at P, $w_1 = r_1^{1/2} e^{i\phi_1/2}$. We now encircle the region along closed curve C through P. Upon travelling anticlockwise once, we have $\phi = \phi_1 + 2\pi$, i.e., $w = r_1^{1/2} e^{i(\phi_1 + 2\pi)/2} = -r_1^{1/2} e^{1\phi_1/2}$ at the point P.

$\Rightarrow$ w = – w_1 at P. This shows that w has changed discontinuously after performing a loop about z = 0, which establishes z = 0 a **Branch Point**.

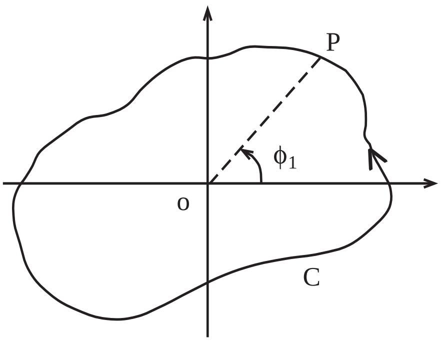


Fig. 32

Now we consider a different loop, a closed curve Γ around some point z* which does not enclose the origin. As before, $z_1 = r_1 e^{i\theta_1}$ and $w_1 = r_1^{1/2} e^{i\phi_1/2}$ upon returning to P, travelling anticlockwise, we have $\phi = \phi_1$ again. Hence w is continuous after performing the loop. So z = z* is not a **Branch Point** for $z^{1/2} = w$.

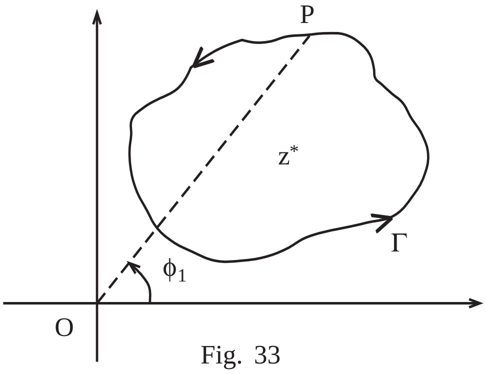


Fig. 33

Example 4. Discuss the multivaluedness of the function $f(z) = (z^2 - 1)^{1/2}$ and introduce cuts to obtain single-valued branches.

Solution : Let $z - 1 = r_1 e^{i\theta}$ and $z + 1 = r_2 e^{i\psi}$

Then $f(z) = \sqrt{r_1 r_2}\, e^{i(\theta+\psi)/2}$

We choose a branch of f(z) at a point z_0 by taking values of θ_0 of θ and ψ_0 of ψ. Then at z_0, f(z) takes the value

$$f_0 = \sqrt{r_1 r_2}\, e^{i(\theta_0+\psi_0)/2}$$

If now z traverses from the point z_0, and form a simple closed contour (end point also z_0) C_0 enclosing the point z = 1, where the point z = –1 lies outside C_0, the value of f(z) at z_0 changes to

$$\sqrt{r_1 r_2}\, e^{i(\theta_0+\psi_0+2\pi)/2} = -f_0$$

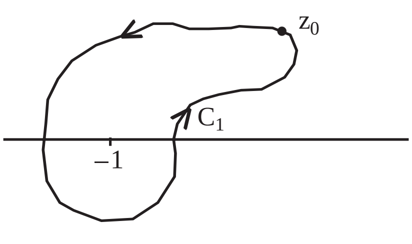


Fig. 34

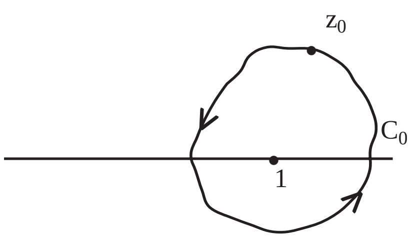


Fig. 35

f(z) takes the same value – f_0 while z travelling from z_0 and returns to z_0 itself forming a closed contour C_1 which encloses –1, but not 1. Hence it is clear that –1 and 1 are the branch points for the function f(z).

In order to obtain single-valued branches we introduce two different set of branch cuts. (i) A branch cut between the points –1 and 1 on the real axis. In this case consider the closed contour C enclosing the branch points –1 and 1. Here f(z) returns to the value (from its value f_0 at z_0).

$$\sqrt{r_1 r_2}\, e^{i(\theta_0+2\pi+\psi_0+2\pi)/2} = \sqrt{r_1 r_2}\, e^{i(\theta_0+\psi_0)/2} = f_0$$

So, it is a single-valued branch.

(ii) Two branch cuts on the real-axis, (–∞, –1) and (1, ∞).

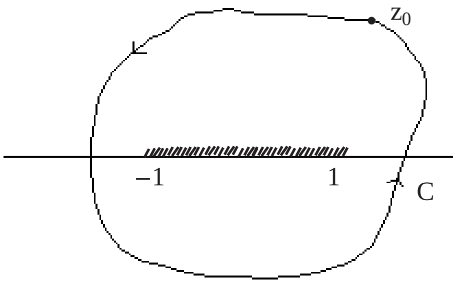


Fig. 36

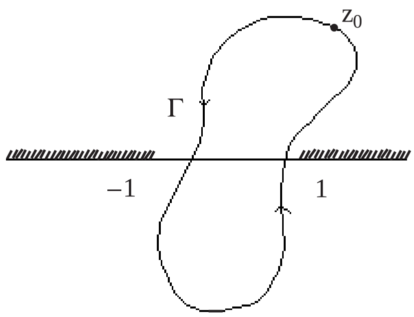


Fig. 37

Here the contour Γ does not enclose any of the branch points, so f(z) remains single-valued as z makes a complete round through Γ initiating from z_0.

Example 5. Construct a branch of log $\left(\frac{z-1}{z+1}\right)$, which is analytic at the origin and takes the values 5πi there.

Solution : Let $g(z) = \log\left(\frac{z-1}{z+1}\right)$. The points z = ± 1 are the branch points of g(z) and the behaviour of g(z) at these branch points are similar to f(z) as shown in the previous example. We do not repeat these here.

Write both z – 1, and z + 1 in polar form :

$z - 1 = re^{i\theta}$, $z + 1 = \rho e^{i\psi}$

Then we can express

$$g(z) = \log\left(\frac{re^{i\theta}}{\rho e^{i\psi}}\right) = \log\left[\frac{r}{\rho} e^{i(\theta-\psi)}\right]$$

$$= \log\left(\frac{r}{\rho}\right) + i(\theta - \psi)$$

We consider the complex z-plane with two branch cuts (–∞, –1), and (1, ∞). Here the principal branch of g(z) is taken as

$$\log\left(\frac{r}{\rho}\right) + i(\theta - \psi), \quad 0 \le \theta < 2\pi; \ -\pi \le \psi < \pi$$

Now, $g_0 = g(0) = i\pi$

In the branch $4\pi \le \theta < 6\pi$; $\pi \le \psi < 3\pi$, g(z) will take the value $5\pi i$ at the origin.

Example 6. Let $z = \omega^2$ and consider Re $\omega > 0$.

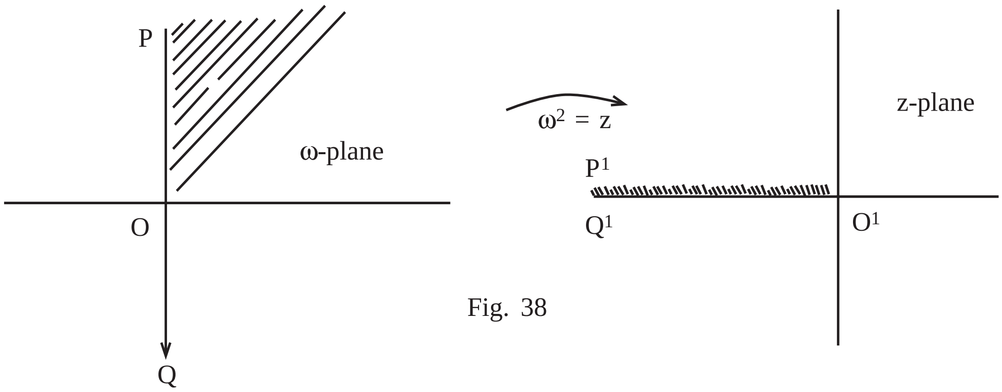


Fig. 38

Image is $z \in \mathbb{C} \setminus (-\infty, 0)$

Note : Injective mapping if Re $\omega > 0$ and $z \in \mathbb{C} \setminus (-\infty, 0)$. We need a **Branch cut** along negative real-axis in the z-plane.

Hence $\quad w = z^{1/2}, \quad z = re^{i\phi}, \quad -\pi < \phi \le \pi$

This ensures that Re $\omega > 0$. Here the points on the cut go either to P or Q. P and Q are arbitrary.

4.5 Integrals of Multi-valued functions

Example 7. Evaluate $\int_0^\infty \frac{x^{\alpha-1}}{1+x} dx, \ 0 < \alpha < 1.$

Let us consider the integal

$$\int_C \frac{z^{\alpha-1}}{1-z} dz$$

where the contour C consists of a large Circle Γ_R with centre at the origin and radius R, a small circle γ_ε with centre origin and radius ε joined to the large circle

Γ_R along the negative side of the real axis from ε to R by means of a cut as shown in the figure 39. Thus we have avoided the branch point z = 0.

We take principal branch of $z^{\alpha-1}$. Then

$$\left|\int_{\Gamma_R} \frac{z^{\alpha-1}}{1-z} dz\right| \le 2\pi R \frac{R^{\alpha-1}}{1+R} = \frac{2\pi R^{\alpha}}{1+R} \to 0 \text{ as } R \to \infty,$$

since $\alpha < 1$,

$$\left|\int_{\gamma_\varepsilon} \frac{z^{\alpha-1}}{1-z} dz\right| \le 2\pi\varepsilon \frac{\varepsilon^{\alpha-1}}{1} = 2\pi\varepsilon^{\alpha} \to 0 \text{ as } \varepsilon \to 0,$$

since $\alpha > 0$.

Thus, by residue theorem,

$$\int_C \frac{z^{\alpha-1}}{1-z} dz = 2\pi i \ \mathrm{Res}\left[\frac{z^{\alpha-1}}{1-z};\ 1\right]$$

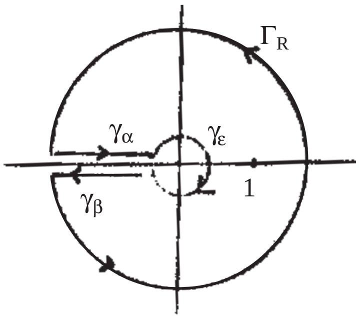


Fig. 39

Observe that $\dfrac{z^{\alpha-1}}{1-z}$ has a simple pole at z = 1 which lies inside C.

or, $$\lim_{R\to\infty}\int_{\Gamma_R} \frac{z^{\alpha-1}}{1-z} dz + \lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} \frac{z^{\alpha-1}}{1-z} dz + \int_{\gamma_\alpha} \frac{z^{\alpha-1}}{1-z} dz + \int_{\gamma_\beta} \frac{z^{\alpha-1}}{1-z} dz = -2\pi i$$

so, $$\int_{\gamma_\alpha} \frac{z^{\alpha-1}}{1-z} dz + \int_{\gamma_\beta} \frac{z^{\alpha-1}}{1-z} dz = -2\pi i \qquad (54)$$

On γ_α : $z = \rho e^{i\pi}$, $0 < \rho < \infty$

so $1 - z = 1 + \rho$ and $dz = e^{i\pi} d\rho$

and $$\int_{\gamma_\alpha} \frac{z^{\alpha-1}}{1-z} dz = \int_{\infty}^{0} e^{i\pi} \frac{\rho^{\alpha-1}}{1+\rho}\ e^{i\pi(\alpha-1)} d\rho = e^{i\pi(\alpha-1)} \int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho} d\rho = -e^{i\pi\alpha}\int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho} d\rho$$

On γ_β, $z = \rho e^{-i\pi}$, $0 < \rho < \infty$

so $1 - z = 1 + \rho$, $dz = e^{-i\pi} d\rho$, then

$$\int_{\gamma_\beta} \frac{z^{\alpha-1}}{1-z} dz + \int_0^\infty e^{-i\pi} \frac{\rho^{\alpha-1}}{1+\rho} e^{-i\pi(\alpha-1)} d\rho = -e^{-i\pi(\alpha-1)} \int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho} d\rho$$

$$= e^{-i\pi\alpha}\int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho}\ d\rho$$

Substituting these integrals into (54), we get

$$[-e^{i\pi\alpha} + e^{-i\pi\alpha}]\int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho} d\rho = -2\pi i$$

i.e. $$\int_0^\infty \frac{\rho^{\alpha-1}}{1+\rho} d\rho = \frac{2\pi i}{2i \sin \pi\alpha}$$

take branch cut on the negative real-axis

or, $$\int_0^\infty \frac{x^{\alpha-1}}{1+x} dx = \frac{\pi}{\sin \pi\alpha}$$

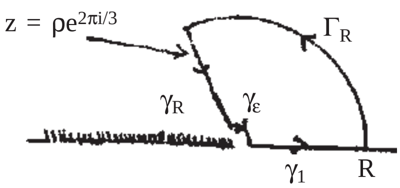


Fig. 40

Example 8 : Evaluate $\int_0^\infty \frac{x^{\alpha-1}}{1+x^3} dx, 0 < \alpha < 3.$

We take the contour integral

$\int_C \frac{z^{\alpha-1}}{1+z^3} dz,$ where C is the contour as shown in the fig. 40. Take principal branch of $z^{\alpha-1}$.

Then, $$\left|\int_{\gamma_\varepsilon} \frac{z^{\alpha-1}}{1+z^3} dz\right| \geq \frac{2\pi}{3}\varepsilon \frac{\varepsilon^{\alpha-1}}{1} = \frac{2\pi}{3}\varepsilon^\alpha \to 0 \text{ as } \to \varepsilon \to 0 \text{ since } \varepsilon > 0$$

and $$\left|\int_{\Gamma_R} \frac{z^{\alpha-1}}{1+z^3} dz\right| \leq \frac{2\pi R}{3} \frac{R^{\alpha-1}}{R^3} = \frac{2\pi}{3} R^{\alpha-3} \to \infty \text{ as } R \to \infty \text{ since } \alpha < 3$$

Now the function $z^{\alpha-1}/_{1+z^3}$ has only one simple pole $z = e^{\frac{i\pi}{3}}$ inside C. Thus

$$\int_C \frac{z^{\alpha-1}}{1+z^3} dz = 2\pi i \ \mathrm{Res}\left[\frac{z^{\alpha-1}}{1+z^3}; e^{i\pi/3}\right] = 2\pi i. \frac{e^{\frac{i\pi}{3}(\alpha-1)}}{3e^{2\pi i/3}} = -\frac{2\pi i}{3} e^{i\alpha\pi/3}$$

i.e., $$\int_{\Gamma_R} \frac{z^{\alpha-1}}{1+z^3} dz + \int_{\gamma_\varepsilon} \frac{z^{\alpha-1}}{1+z^3} + \int_R^\varepsilon \frac{\rho^{\alpha-1}}{1+\rho^3} e^{2\pi i(\alpha-1)/3} e^{2\pi i/3} d\rho + \int_\varepsilon^R \frac{\rho^{\alpha-1}}{1+\rho^3} d\rho = -2\pi i \frac{e^{\alpha\pi i/3}}{3}$$

[In the third integral, we used $z = \rho e^{2\pi i/3}$, $dz = e^{2\pi i/3} d\rho$, $1 + z^3 = 1 + \rho^3$, and in the fourth integral, $z = \rho$, $dz = d\rho$]

Taking $R \to \infty$ and $\varepsilon \to 0$ in the above integrals, we find using the earlier results

$$-e^{2\alpha\pi i/3}\int_0^\alpha \frac{\rho^{\alpha-1}}{1+\rho^3} d\rho + \int_0^\alpha \frac{\rho^{\alpha-1}}{1+\rho^3} d\rho = \frac{2\pi i e^{\alpha\pi i/3}}{3}$$

So that, $$\int_0^{\infty} \frac{\rho^{\alpha-1}}{1+\rho^3} d\rho = \frac{2\pi i}{3} \cdot \frac{1}{e^{\alpha\pi i/3} - e^{-\alpha\pi i/3}} = \frac{\pi}{3\sin\frac{\alpha\pi}{3}}$$

or, $$\int_0^{\infty} \frac{x^{\alpha-1}}{1+x^3} dx = \frac{\pi}{3\sin\frac{\alpha\pi}{3}}$$

Riemann Surface

A Riemann surface is a generalization of the complex plane to a surface comprising several sheets so that a multi-valued function can have only one value corresponding to each point on that surface. Once such a surface is ascertained for a given multi-valued function, the function becomes single-valued on the surface and can be treated according to the theory of single-valued functions.

This topology removes artificial restrictions-**Branch Cuts** and gives us a more general notion of a domain so that a multi-valued analytic function becomes single-valued if it is considered as a mapping to an appropriate generalized domain as suggested by G. F. B. Riemann (1826-1866) in 1851. The idea is ingenious—a geometric construction that permits surfaces to be the domain or range of a multi-valued function.

4.6 Branch points at infinity

So far we have considered only branch points in the finite plane. Now we discuss about the possibility of a branch point at infinity. For this sake we map the point at infinity to the origin with the transformation $\varsigma = 1/z$ and then examine the point $\varsigma = 0$.

Example 9 : Again we consider the multi-valued function $f(z) = z^{1/2}$. Making the transformation $\varsigma = \frac{1}{z}$, the point at infinity is mapped to the origin, we have $f(\varsigma) = \left(1/\varsigma\right)^{1/2}$. For each value of ς, there are two values of $\varsigma^{-1/2}$. Writing $\varsigma^{-1/2}$ in modulus-argument form

$$\varsigma^{-1/2} = \frac{1}{\sqrt{|\varsigma|}} e^{-i\mathrm{Arg}(\varsigma)/2}$$

we find that like $z^{1/2}$, $\varsigma^{-1/2}$ possesses double sheeted Riemann surface. We see that each time we walk around the origin, the argument of $\varsigma^{-1/2}$ changes by $-\pi$. This means that the value of the function changes by the factor $e^{-i\pi} = -1$, i.e. the function changes sign. If we walk around the origin twice, the argument changes by -2π, so that the value of the function does not change, $e^{-2\pi i} = 1$.

Now, since $\varsigma^{-1/2}$ has a branch point at zero, we conclude that $z^{1/2}$ has a branch point at infinity.

Example 10 : Again consider the multi-valued logarithm function $f(z) = \log z$. Mapping the point at infinity to the origin, we have

$$f(\varsigma) = \log\left(\frac{1}{\varsigma}\right) = -\log \varsigma$$

But $\log \varsigma$ has a branch point at $\varsigma = 0$. Thus $\log z$ has a branch point at infinity.

Branch points at infinity : Paths around infinity

We can also check for a branch point at infinity by considering a path that encloses the point at infinity and no other singularities. This can be done by drawing a simple closed curve that separates the complex plane into a bounded region that contains all the singularities of the function in the finite plane. Then, depending upon the orientation, the curve is a contour enclosing all the finite singularities, or the point at infinity and no other singularities.

Once again consider the function $z^{1/2}$. We know that the function changes value on a curve that goes around the origin. Such a curve can be considered to be either a path around the origin or a path around the point at infinity. In either case the path encloses one branch point. Now consider a curve that does not go around the origin. Such a curve can be considered to be either a path around neither of the branch points or both of them. Thus we see that $z^{1/2}$ does not change value when we follow a path that encloses neither or both of its branch points.

Example 11 : Consider the multi-valued function $f(z) = (z^2 - 1)^{1/2}$. Rewriting the function $f(z) = (z - 1)^{1/2} (z + 1)^{1/2}$, we see that there are branch points at $z = \pm 1$. Now consider the point at infinity.

$$f(\varsigma^{-1}) = (\varsigma^{-2} - 1)^{1/2} = \pm\varsigma^{-1}(1 - \varsigma^2)^{1/2}$$

which shows that $f(\varsigma^{-1})$ does not have a branch point at $\varsigma = 0$ and $f(z)$ does not have a branch point at infinity. We might reach the same conclusion by considering a path around the point at infinity. Consider a path that encircles the branch points at $z = \pm 1$ once in the positive direction. Equivalently it encircles the point at infinity once in the negative direction. In traversing this path, the value of $f(z)$ is multiplied by the factor $(e^{2i\pi})^{1/2} (e^{2i\pi})^{1/2} = e^{2i\pi} = 1$. Thus the value of the function remains unchanged. There is no branch point at infinity.

4.7 Detection of branch points

We have the definition of a branch point, but we do not have a convenient criterion for determining if a particular function has a branch point. We have noticed that log z and z^k for non-integer k have branch points at zero and infinity. The inverse trigonometric functions like $\sin^{-1}z$, $\cos^{-1}z$ etc. also have branch points, but they can be written in terms of the logarithm and the square root. In fact all the elementary functions with branch points can be written in terms of the functions log z and z^k. Furthermore, note that the multi-valuedness of z^k comes from the logarithm, $z^k = e^{k\log z}$. This gives us a way of determining branch points of a function if there is any.

Result : Let f(z) be a single-valued function. Then log f(z) and $(f(z))^k$ may have branch points only where f(z) is zero or singular.

Example 12 : Consider the functions

1. $(z^2)^{1/2}$ 2. $(z^{1/2})^2$ 3. $(z^{1/2})^3$

Are they multi-valued? Do they have branch points?

Solution

1. $$\left(z^2\right)^{1/2} = \pm\sqrt{z^2} = \pm z$$

Because of $(\cdot)^{1/2}$, the function is multi-valued. The only possible branch points are at zero and point at infinity. If $((e^{i\theta})^2)^{1/2} = 1$, then as $((e^{2\pi i})^2)^{1/2} = (e^{4\pi i})^{1/2} = e^{2\pi i} = 1$ the function does not change value when we walk around the origin. We can also consider this to be a path around infinity. This function is multi-valued, but has no branch points.

2. $$\left(z^{1/2}\right)^2 = \left(\pm\sqrt{z}\right)^2 = z$$

This function is single-valued.

3. $$\left(z^{1/2}\right)^3 = \left(\pm\sqrt{z}\right)^3 = \pm\left(\sqrt{z}\right)^3$$

This function is multi-valued. We consider the possible branch point at z = 0. If $(e^{i0})^{1/2})^3 = 1$, then as $((e^{2i\pi})^{1/2})^3 = ((e^{i\pi 2})^{1/2})^3 = (e^{i\pi})^3 = e^{3\pi i} = -1$, the function changes value when we walk around the origin. So it has a branch point at z = 0. Since this is also a path around infinity, there is a branch point at the point at infinity.

Example 13 : Consider the function f(z) = log (1/z – 1). Since $\frac{1}{z-1}$ has only zero at infinity and its only singularity (a pole here) is at z = 1, the only, possible branch points are at z = 1 and z = ∞.

Here $f(z) = \log\left(\frac{1}{z-1}\right) = -\log(z-1) = \log\omega$, say

We know that log ω has branch points at zero and infinity, so f(z) has branch points at z = 1 and z = ∞.

Example 14 : Consider the functions

1. $e^{\log z}$ 2. $\log e^z$

Are they multi-valued? Do they have branch points?

Solution :

1. $e^{\log z} = e^{\log z + i2\pi k}$, k = 0, ± 1, ...

$= e^{\text{Log} z} e^{i2\pi k} = z$

The function is single-valued.

2. $\log e^z = \text{Log} e^z + i2\pi k = z + i2\pi k$, k = 0, ± 1, ...

This function is multi-valued. It may have branch points only where e^z is zero or infinite. This occurs only at z = ∞. Thus there are no branch points in the finite plane. The function does not change when traversing a simple closed path and since this path can be considered to enclose the point at infinity, there is no branch point at infinity.

Note : Let f(z) be single-valued and have either a zero or a singularity at $z = z_0$. Then $\{f(z)\}^k$ may have a branch point at $z = z_0$. If f(z) is not a power of z, then we are not sure whether $\{f(z)\}^k$ changes value when we walk around z_0.

Now if f(z) can be decomposed into factors f(z) = h(z) g(z), where h(z) is finite and non zero at z_0, then from g(z) we know how fast f(z) vanishes or tends to infinity. Again $\{f(z)\}^k = \{h(z)\}^k \{g(z)\}^k$ and $\{h(z)\}^k$ does not have a branch point at z_0. So that $\{f(z)\}^k$ has a branch point at z_0 if and only if $\{f(z)\}^k$ has a branch point there.

Similarly, we can decompose

$$\log \{f(z)\} = \log \{h(z)g(z)\} = \log \{h(z)\} + \log \{g(z)\}$$

to see that log {f(z)} has a branch point at z_0 if and only if log {g(z)} has a branch point there.

Example 15 : Consider the functions :

1. $\sin z^{1/2}$ 2. $(\sin z)^{1/2}$ 3. $z^{1/2} \cos z^{1/2}$ 4. $(\sin z^2)^{1/2}$.

Find the branch points and the number of branches.

Solution : 1. $\sin z^{1/2} = \sin\left(\pm\sqrt{z}\right) = \pm \sin\sqrt{z}$

So it is multi-valued. It has two branches and the possible branch points are zero and infinity. Consider the unit circle |z| = 1 which is a path around the origin and infinity. If

$\sin(e^{i0})^{1/2} = \sin(1)$, then as

$$\sin((e^{i2\pi})^{1/2}) = \sin(e^{i\pi}) = \sin(-1) = -\sin 1,$$

there are branch points at the origin and infinity

2. $(\sin z)^{1/2} = \pm\sqrt{\sin z}$

The function is multi-valued and has two branches. The sine function vanishes at $z = n\pi$ and is singular at infinity. These may be branch points of the function. Consider the point $z = n\pi$. We can express

$$\sin z = (z - n\pi)\frac{\sin z}{z - n\pi}, \text{ n an integer.}$$

But $$\lim_{z\to n\pi}\frac{\sin z}{z - n\pi} = \lim_{z\to n\pi}\frac{\cos z}{1} = (-1)^n$$

So, $(\sin z)^{1/2}$ has branch points at $z = n\pi$ since $(z - n\pi)^{1/2}$ has a branch point at $z = n\pi$.

Here the branch points are $z = n\pi$, $n = 0, \pm1, \ldots$ and they go to infinity. So it is not possible to make a path that encloses infinity and no other singularities. The point at infinity is a non-isolated singularity. A point can be a branch point only if it is an isolated singularity.

3. $z^{1/2} \cdot \cos z^{1/2} = \pm\sqrt{z}\cos\left(\pm\sqrt{z}\right)$

$$= \pm\sqrt{z}\cos\sqrt{z}$$

The function is multi-valued. It may possess branch points at $z = 0$ and $z = \infty$. If $(e^{i0})^{1/2}\cos(e^{i0})^{1/2} = \cos(1)$, then as $(e^{i2\pi})^{1/2}\cos((e^{i2\pi})^{1/2}) = (-1)\cos(e^{i\pi}) = -\cos(-1) = -\cos 1$, there are branch points at the origin and infinity.

4. $(\sin z^2)^{1/2} = \pm\sqrt{\sin z^2}$

The function is multi-valued. Now since siz $z^2 = 0$ at $z = (n\pi)^{1/2}$, there may be branch points there.

We consider first the point $z = 0$. We can write

$$\sin z^2 = z^2\,\frac{\sin z^2}{z^2}$$

but $$\lim_{z\to0}\frac{\sin z^2}{z^2} = \lim_{z\to0}\frac{2z\cos z^2}{2z} = 1$$

So, $(\sin z^2)^{1/2}$ does not have a branch point at $z = 0$ as $(z^2)^{1/2}$ does not have a branch point there.

Next consider the point $z = \sqrt{n\pi}$

$$\sin z^2 = \left(z - \sqrt{n\pi}\right)\frac{\sin z^2}{z - \sqrt{n\pi}}$$

but $$\lim_{z\to\sqrt{n\pi}} \frac{\sin z^2}{z - \sqrt{n\pi}} = \lim_{z\to\sqrt{n\pi}} \frac{2z\cos z^2}{1} = 2\sqrt{n\pi}(-1)^n$$

Since $\left(z - \sqrt{n\pi}\right)^{1/2}$ has a branch point at $z = \sqrt{n\pi}, (\sin z^2)^{1/2}$, too as a branch point there.

Thus we see that $(\sin z^2)^{1/2}$ has branch points at $z = (n\pi)^{1/2}$ for $n \in Z \setminus \{0\}$. This is the set of numbers : $\left\{\pm\sqrt{\pi}, \pm\sqrt{2\pi}, \ldots, \pm i\sqrt{\pi}, \pm i\sqrt{2\pi}, \ldots\right\}$. The point at infinity is a non-isolated singularity and hence it is not included in the set of branch points.

Example 16 : Find the branch points of

$$f(z) = (z^3 - z)^{1/3}$$

and introduce the branch cuts. If $f(3) = 2\sqrt[3]{3}$, find $f(-3)$.

Solution : Here $f(z) = z^{1/3}(z - 1)^{1/3} (z + 1)^{1/3}$

So the branch points are at $z = -1$, 0 and 1. We consider the point at infinity

$$f\left(1/\varsigma\right) = \left(\frac{1}{\varsigma}\right)^{1/3}\left(\frac{1}{\varsigma} - 1\right)^{1/3}\left(\frac{1}{\varsigma} + 1\right)^{1/3}$$
$$= \frac{1}{\varsigma}(1 - \varsigma)^{1/3}(1 + \varsigma)^{1/3}$$

Since $f(1/\varsigma)$ does not have a branch point at $\varsigma = 0$, $f(z)$ does not have a branch point at infinity.

Here we give three possible branch cuts :

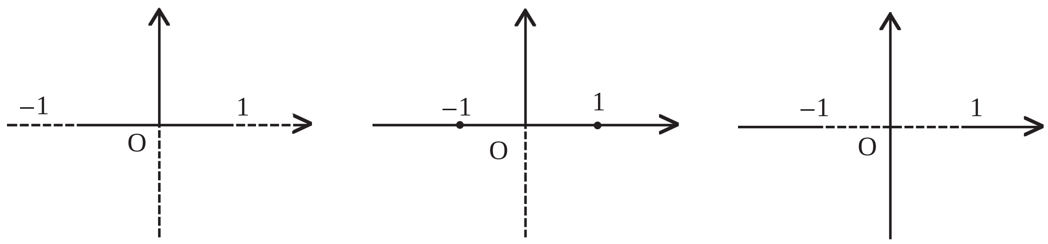


Fig. 41 Three possible branch cuts for $f(z) = (z^3 - z)^{1/3}$

In the first and third the function is single-valued but in the second it is not. It is clear that the first branch cut does not allow us to walk around any of the branch points.

The second branch cut allows us to walk around the branch points at z = ± 1. If we walk around these two once in the positive direction, the value of the function would change by the factor $e^{i4\pi/3}$.

The third branch cut allows us to walk around all the three branch points, the value of the function will not change (since $e^{i6\pi/3} = e^{i2\pi} = 1$).

To find f(–3), we consider the third branch cut with $f(3) = 2\sqrt[3]{3}$.

$$f(3) = (3e^{i0})^{1/3}(2e^{i0})^{1/3}(4e^{i0})^{1/3} = 2\sqrt[3]{3}$$

The value of f(–3) is

$$f(-3) = (3e^{i\pi})^{1/3}(2e^{i\pi})^{1/3}(4e^{i\pi})^{1/3} = -\,2\sqrt[3]{3}$$

Example 17 : Determine the branch points of the function $f(z) = (z^3 - 1)^{1/2}$.

Construct branch cuts and define a branch so that z = 0 and z = –1 do not lie on a cut, such that f(0) = –i; then what is f(–1/2)?

Solution : The roots of the equation $z^3 - 1 = 0$ are

$$\left\{1, e^{i2\pi/3}, e^{i4\pi/3}\right\} = \left\{1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\right\}$$

so that,

$$\left(z^3 - 1\right)^{1/2} = (z-1)^{1/2}\left(z + \frac{1-i\sqrt{3}}{2}\right)^{1/2}\left(z + \frac{1+i\sqrt{3}}{2}\right)^{1/2}$$

There are branch points at each of the cube roots of unity

$$z = \left\{1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\right\}$$

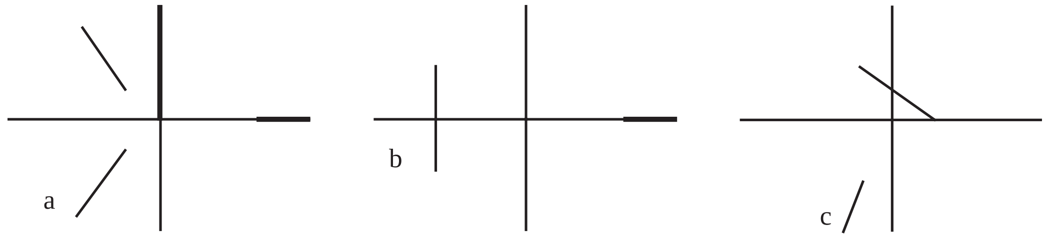
Now we examine the point at infinity. We make the change of variable $z = 1/\varsigma$

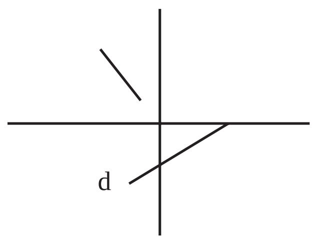
$$f(1/\varsigma) = (1/\varsigma^3 - 1)^{1/2} = \varsigma^{-3/2}(1 - \varsigma^3)^{1/2}$$

$\varsigma^{-3/2}$ has a branch point at $\varsigma = 0$, while $(1 - \varsigma^3)^{1/2}$ is not singular there. Since $f(1/\varsigma)$ has a branch point at $\varsigma = 0$, f(z) has a branch point at infinity.

There are several ways of introducing branch cuts to separate the branches of the function. The easiest approach is to put a branch cut from each of the three branch points in the finite complex plane out to the branch point at infinity (see Figure 42a). Clearly this makes the function single-valued as it is impossible to walk around any of the branch points. Another approach is to have a branch cut from one of the branch points in the finite plane to the branch point at infinity and a branch cut connecting the remaining two branch points (see Figure 42 bcd). In this case, in walking around any one of the finite branch points (in the + ve direction), the argument of the function changes by π. This means that the value of the function changes by $e^{i\pi}$, which is to say, the value of the function changes sign. In walking around any two of the finite branch points (in the +ve direction), the argument of the function changes by 2π i.e., the value of the function changes by $e^{i2\pi}$, that means the value of the function does not change.

Figure 42. Branch cuts for $(z^3-1)^{1/2}$





Now we choose the branch 42a, and introduce the variables $z - 1 = r_1 e^{i\theta}$, $0 \le \theta_1 < 2\pi$

$$z + \frac{1 - i\sqrt{3}}{2} = r_2 e^{i\theta_2}, \quad -\frac{2\pi}{3} \le \theta_2 < \frac{\pi}{3}$$

$$z + \frac{1 - i\sqrt{3}}{2} = r_3 e^{i\theta_3}, \quad -\frac{\pi}{3} \le \theta_3 < \frac{2\pi}{3}$$

We compute f(0) to see whether it has the desired value,

$$f(z) = \sqrt{r_1 r_2 r_3}\, e^{i(\theta_1+\theta_2+\theta_3)/2}$$

$$f(0) = e^{i(\pi - \pi/3 + \pi/3)/2} = e^{i\pi/2} = i$$

Since it does not have the desired value, we change the range of θ_1,

$$z - 1 = r_1 e^{i\theta_1}, \; 2\pi \le \theta_1 < 4\pi$$

f(0) now has the desired value,

$$f(0) = e^{i(3\pi - \pi/3 + \pi/3)} = -\, i$$

We compute $f\left(-\frac{1}{2}\right)$,

$$f\left(-\frac{1}{2}\right) = \sqrt{\frac{3}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}}\; e^{i\left(3\pi - \frac{\pi}{2} + \frac{\pi}{2}\right)/2}$$

$$= \sqrt{\frac{9}{8}} e^{i3\pi/2} = \frac{-3i}{2\sqrt{2}}$$

Example 18 : Identify the branch points of the function

$$\omega = f(z) = (z^3 + z^2 - 6z)^{1/2}$$

in the extended complex plane. Specify the branch cuts and select a branch that gives a single-valued function where it is continuous at $z = -1$ with $f(-1) = -\sqrt{6}$.

Solution : First we factor the function

$$f(z) = [z(z - 2(z + 3)]^{1/2} = z^{1/2}(z - 2)^{1/2} (z + 3)^{1/2}$$

There are branch points at $z = -3, 0, 2$. Now we examine the point at infinity.

$$f(1/\varsigma) = \left[\frac{1}{\varsigma}\left(\frac{1}{\varsigma} - 2\right)\left(\frac{1}{\varsigma} + 3\right)\right]^{1/2} = \varsigma^{-3/2}\left[(1 - 2\varsigma)(1 + 3\varsigma)\right]^{1/2}$$

Since $\varsigma^{-3/2}$ has a branch point at $\varsigma = 0$ and the rest of the terms are analytic there, f(z) has a branch point at infinity.

Now consider the branch cuts given in the figure 43. These cuts do not permit us to walk around any single branch point. We can walk around none of the branch points (or all of the branch points considering the cuts [–3, 2] and $x = 0$, $y \le 0$). The cuts can be used to define a single-valued branch of the function.

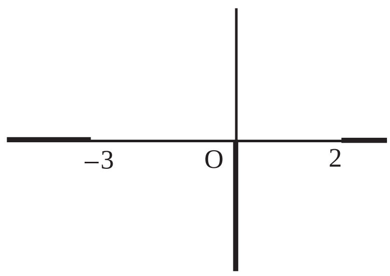


Fig. 43

Now to define the branch, we choose $z + 3 = r_i\ e^{i\theta_1}$, $-\pi \le \theta_1 < \pi$; $z = r_2 e^{i\theta_2}$, $\frac{-\pi}{2} \le \theta_2 < \frac{3\pi}{2}$ and $z - 2 = r_3 e^{i\theta_3}$, $0 \le \theta_3 < 2\pi$.

The function is, $f(z) = (r_1 r_2 r_3)^{1/2} e^{i(\theta_1 + \theta_2 + \theta_3)/2}$

Here $f(-1) = [(2)(1)(3)]^{1/2} e^{i(0 + \pi + \pi)/2} = -\sqrt{6}$

So our choice of angles gave the desired branch.

4.8 The Riemann surface for $\omega = z^{1/2}$

We have seen that $\omega = z^{1/2}$ possesses two branch points $z = 0$ and $z = \infty$. To utilize the developments made in Example 1, we introduce a branch cut along the negative real axis. The given function has two values for any $z \ne 0$.

$$f_1(z) = r^{1/2} e^{i\theta/2}, \quad -\pi < \theta \le \pi$$

and $f_2(z) = r^{1/2} e^{i\theta/2}, \pi < \theta \leq 3\pi$

Each function f_1 and f_2 is single-valued on the domain formed by cutting the z-plane along the negative real-axis. Let D_1 and D_2 be the domains of f_1 and f_2 respectively. The range set for f_1 is the set R_1 consisting of the right-half plane and the positive imaginary axis [see Figure 28b]; the range set for f_2 is the set R_2 consisting of the left-half plane and the negative imaginary axis [see Figure 29b]. The sets R_1 and R_2 are glued together along the positive imaginary axis and the negative imaginary axis to form the w-plane with the origin deleted. We stack D_1 directly above D_2. The edge of D_1 in the upper-half plane is joined to the edge of D_2 in the lower-half plane, and the edge of D_1 in the lower-half plane is joined to the edge of D_2 in the upper-half plane (it is needless to mention that the line of joining is the negative real-axis). When these domains are glued together in this manner, they form a Riemann surface domain for the mapping $w = f(z) = z^{1/2}$ shown in the figure 44 for some finite r.

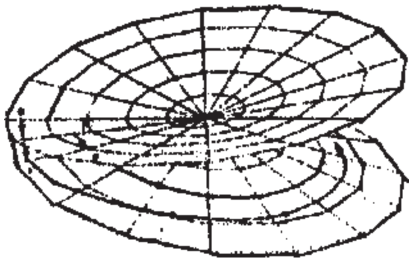


Fig. 44

4.9 Concept of neighbourhood

When a point lies on the boundary of two domains D_1 and D_2, we define a neighbourhood of that point in the following manner. Here the boundary of D_1 and D_2 is the negative real-axis. (i) Neighbourhood of a point $\varsigma \in D_1$ with Im $\varsigma < 0$, Arg $\varsigma = \pi$, $|z - \varsigma| < \varepsilon$ consists of points on : (a) D_1 if Im $\varsigma \geq 0$ (b) D_2 if Im $\varsigma < 0$. (ii) Neighbourhood of a point $\eta \,\varepsilon\, D_2$ with Im $\eta = 0$, Arg $\eta = 3\pi$, $|z - \eta| < \varepsilon$ consists of points on (a) D_1 if Im $\eta < 0$ and (b) D_2 if Im $\eta \geq 0$. With these definitions of neighbourhood of a point, it becomes clear that $w = z^{1/2}$ is continuous and differentiable everywhere on the Riemann surface except at the origin and the point at infinity. The derivative is given by

$$\frac{d}{dz} z^{1/2} = \begin{cases} \frac{1}{2} \frac{1}{f_1} \text{ on } D_1 \\ \frac{1}{2} \frac{1}{f_2} \text{ on } D_2 \end{cases}$$

4.10 The Riemann Surface for w = log z

The Riemann surface for the multivalued function $\omega = \log z$ is similar to the one we presented for the square root function. However, it requires infinitely many copies of the z-plane cut along the negative x-axis, which mark S_k for k = ..., –n, ..., –1, 0, 1, ..., n, ... Now we stack these cut planes directly on each other so that the corresponding points have the same position. We join the sheet S_k to S_{k+1} as follows. For each integer k, the edge of the sheet S_k in the upper half-plane is joined to the edge of the sheet S_{k+1} in the lower-half plane. The Riemann surface for the domain of log z looks like a spiral staircase that extends upward on the sheets S_1, S_2..., and downward on the sheets S_{-1}, S_{-2}, ... as shown in figure 45. For S_k, we use

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta), \text{ where}$$
$$r = |z| \text{ and } 2\pi k - \pi < \theta \le \pi + 2\pi k$$

Again, for S_k, the correct branch of log z on each sheet is

$$\log z = \log r + i\theta, \text{ where}$$
$$r = |z| \text{ and } 2\pi k - \pi < \theta \le \pi + 2\pi k$$

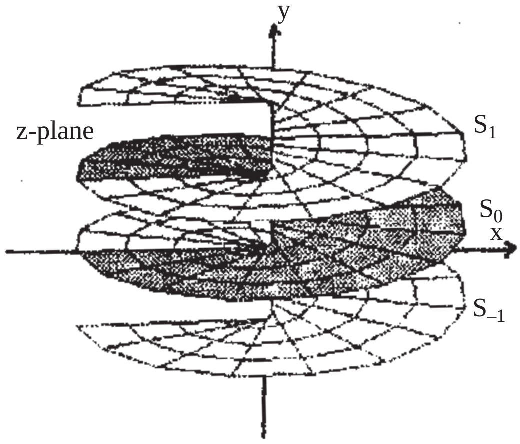


Fig. 45

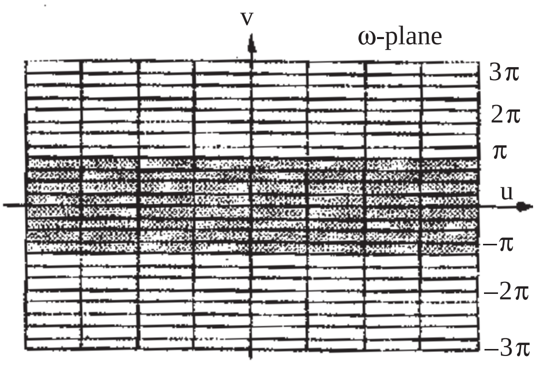


Fig. 46

Example 19 : Form a Riemann surface for $f(z) = (z - 1)^{1/3}$ taking a branch cut along the line y = 0, x ≥ 1. Detect the point where the function takes the value $\sqrt{2}\,(i - 1)$.

Solution : Let $r = |z - 1|$ and $\theta = \arg(z - 1)$, where $0 \le \theta < 2\pi$. Here the Riemann surface consists of three domains D_1 D_2 and D_3 :

$$f_1(z) = r^{1/3}e^{i\theta/3}, \quad 0 \le \theta < 2\pi \ (D_1)$$
$$f_2(z) = r^{1/3}\, e^{i\theta/3}, \quad 2\pi \le \theta < 4\pi \ (D_2)$$

$$f_3(z) = r^{1/3}\, e^{i\theta/3}, \quad 4\pi \le \theta < 6\pi \ (D_3)$$

Each function f_1, f_2 and f_3 is single-valued on the domain formed by cutting the z-plane along the line $y = 0$, $x \ge 1$.

We place D_1 on the top, then D_2 and D_3. The edge of D_1 in the upper-half plane is joined to the edge of D_2 in the lower-half plane and the edge of D_2 in the upper-half plane is joined to the edge of D_3 in the lower-half plane and finally the edge of D_3 in the upper-half plane is joined to the edge of D_1 in the lower-half plane.

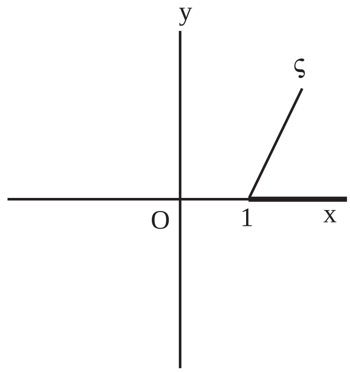


Fig. 47

Say at $z = \varsigma$, $f(\varsigma) = \sqrt{2}\,(i - 1)$

i.e. $$f(\varsigma) = -2\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

$$= 2e^{i\pi} e^{-\frac{i\pi}{4}} = 2e^{i3\pi/4}$$

$$= 2e^{i\left(\frac{9\pi}{4}\right)/3} = 2e^{i\left(\frac{\pi}{4}+2\pi\right)/3}$$

So, $\varsigma - 1 = 2^3 e^{\frac{i\pi}{4}}, \varsigma = 1 + 8e^{\frac{i\pi}{4}}$ lying in the domain D_2.

Example 20 : Form the Riemann surface for the function $f(z) = (z^2 - 1)^{1/2}$.

Solution : Here the given function $f(z) = (z^2 - 1)^{1/2}$ has branch points at $z = \pm 1$. To examine the point at infinity, we substitute $z = 1/\varsigma$ and examine the point $\varsigma = 0$.

$$f\left(\frac{1}{\varsigma}\right) = \left[\left(\frac{1}{\varsigma}\right)^2 - 1\right]^{i/2} = \frac{1}{(\varsigma^2)^{1/2}}(1 - \varsigma^2)^{1/2}$$

Since there is no branch point at $\varsigma = 0$, $f(z)$ has no branch point at infinity.

Let $z - 1 = r_1 e^{i\phi_1}$ and $z + 1 = r_2 e^{i\phi_2}$,

where $\phi_1 = \text{Arg}\,(z - 1)$ and $\phi_2 = \text{Arg}\,(z + 1)$. Then $\omega = f(z) = (z^2 - 1)^{1/2}$

$$= (z - 1)^{1/2}\,(z + 1)^{1/2} = (r_1 r_2)^{1/2}\, e^{i(\phi_1 + \phi_2)}$$

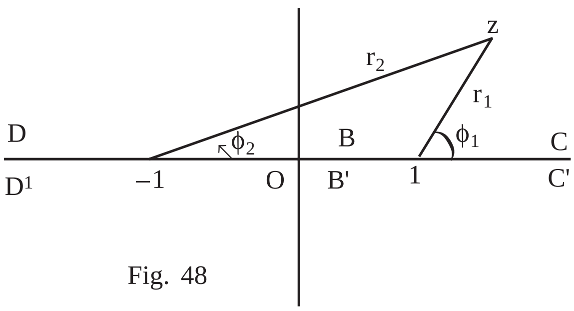


Fig. 48

Case-I $0 \le \phi_1 < 2\pi,\ 0 \le \phi_2 < 2\pi$

on the segment	ϕ_1	ϕ_2	$e^{i(\phi_1+\phi_2)/2}$	Continuity of f(z)
B	π	0	i	No
B′	π	2π	–i	
C	0	0	1	Yes
C′	2π	2π	1	
D	π	π	–1	Yes
D′	π	π	–1	

Fig. 49

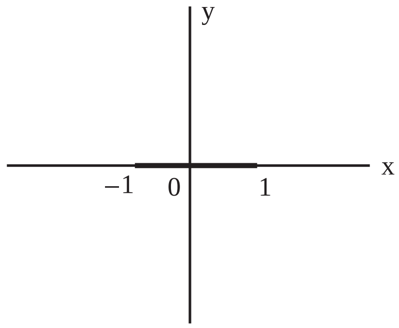


Fig. 50 Branch cut [–1, 1]

Case-II $0 \le \phi_1 < 2\pi,\ -\pi \le \phi_2 < \pi$

on the segment	ϕ_1	ϕ_2	$e^{i(\phi_1+\phi_2)/2}$	Continuity of f(z)
B	π	0	i	Yes
B′	π	0	i	
C	0	0	1	No
C′	2π	0	–1	
D	π	π	–1	No
D′	π	$-\pi$	1	

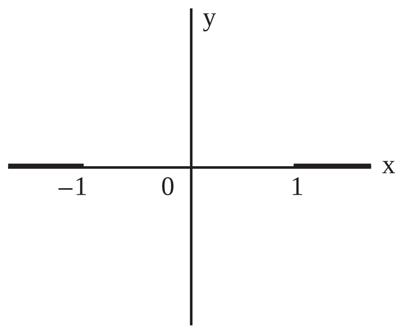


Fig. 51 Branch cuts (–∞, –1] and [1, ∞)

Two branches of $(z-1)^{1/2}$ can be taken as

$$f_1(z) = \sqrt{r_1}e^{i\phi_1/2} \text{ and } f_2(z) = \sqrt{r_1}e^{i(\phi_1+2\pi)/2},\ 0 \le \phi_1 < 2\pi = -f_1(z)$$

Again two branches of $(z+1)^{1/2}$ can be taken as

$$g_1(z) = \sqrt{r_2}e^{i\phi_2/2} \text{ and } g_2(z) = \sqrt{r_2}e^{i(\phi_2+2\pi)/2},\ 0 \le \phi_2 < 2\pi$$
$$= -\, g_1(z)$$

Let us construct a Riemann surface for $\omega = (z^2 - 1)^{1/2}$ considering case I.

Here a Riemann surface consists of two sheets So and S_1. Let S_0 be an extended complex plane cut along the real axis from z = –1 to z = 1 and S_1 be another extended complex plane cut of similar nature.

$$S_0\begin{cases}0 \le \operatorname{Arg}(z-1) < 2\pi \\ 0 \le \operatorname{Arg}(z+1) < 2\pi\end{cases} \quad S_1\begin{cases}2\pi \le \operatorname{Arg}(z-1) < 4\pi \\ 2\pi \le \operatorname{Arg}(z+1) < 4\pi\end{cases}$$

The sheets S_0 and S_1 are joined along the segment [–1, 1] in such a way that the lower edge of the slit in S_0 is joined to the upper edge of the slit in S_1, and the lower edge of the slit in S_1 is joined to the upper edge of the slit in S_0.

Let a point on the sheet S_0 move anticlockwise and form a simple closed curve which encloses the segment [–1, 1] once. Then both ϕ_1 and ϕ_2 change by an amount 2π upon returning to their original position. i.e. $(\phi_1 + \phi_2)/2$ changes by an amount 2π, so the value of

$$\omega = (r_1 r_2)^{1/2} e^{i(\phi_1+2\pi+\phi_2+2\pi)/2} = (r_1 r_2)^{1/2} e^{i(\phi_1+\phi_2)/2}$$

remains unchanged.

Then $\omega = f_1 g_1$ on S_0 and as well as on S_1.

If a point starting on the sheet S_0 travels a path which makes a complete round about only the branch point $z = 1$, it crosses from the sheet S_0 to S_1. In this case, the value of ϕ_1 changes by an amount 2π, while the value of ϕ_2 does not change at all. The change in $(\phi_1 + \phi_2)/2$ is then π. The change in $(\phi_1 + \phi_2)/2$ remains the same if a point on the sheet S_0 makes a complete round about the branch point $z = -1$ only and enters on the S_1 sheet. This time.

$$\omega = \begin{cases} f_1 g_1 \text{ on } S_0 \\ -f_1 g_1 \text{ on } S_1 \end{cases}$$

Thus the double-valued function $\omega = (z^2 - 1)^{1/2}$ can now be considered as a single-valued function on the Riemann surface constructed above. Hence the transformation $\omega = (z^2 - 1)^{1/2}$ maps each of the sheets S_0 and S_1 forming the Riemann surface on the entire ω-plane.

Riemann surface for the case II

Here the Riemann surface is formed by two sheets S_0 and S_1. Each of these sheets is an extended complex plane cut along the line $(-\infty, -1) \cup [1, \infty)$

$$S_0 \begin{cases} 0 \le \text{Arg}(z-1) < 2\pi \\ -\pi \le \text{Arg}(z+1) < \pi \end{cases} \quad S_1 \begin{cases} 2\pi \le \text{Arg}(z-1) < 4\pi \\ \pi \le \text{Arg}(z+1) < 3\pi \end{cases}$$

These sheets are joined along the line $(-\infty, -1] \cup [1, \infty)$ in such a way that the lower edge of the slit in S_0 is joined to the upper edge of the slit in S_1, and the lower edge of the slit in S_1 is joined to the upper edge of the slit in S_0.

If a point traverses a simple closed curve on either of the sheets S_0 or S_1 not enclosing any of the branch points –1 or 1, then the function f(z) remains single-valued on the respective sheet, whereas if it encloses any one of the branch points the function changes the branch as explained in case I. In the same way the double-valued function $f(z) = (z^2 - 1)^{1/2}$ can be treated as a single-valued function on the Riemann surface formed earlier.

Example 21 : The power function $\omega = f(z) = [z(z-1)(z-2)]^{1/2}$ has two branches. Show that $f(-1)$ can be either $-\sqrt{6}i$ or $\sqrt{6}i$. Suppose the branch that corresponds to $f(-1) = -\sqrt{6}i$ is chosen, find the value of the function at $z = i$.

Solution : The given power function can be expressed as

$$\omega = f(z) = \sqrt{|z(z-1)(z-2)|}\, e^{i[\operatorname{Arg} z + \operatorname{Arg}(z-1) + \operatorname{Arg}(z-2)]/2} e^{ik\pi},\ k = 0,\ 1$$

where the two possible values of k correspond to the two branches of the double-valued power function.

If figure 52a branch cuts are $y = 0,\ x \le 0$ and $y = 0,\ 1 \le x \le 2$ and in figure 52b branch cuts are $y = 0,\ 0 \le x \le 1$ and $y = 0,\ x \ge 2$. In both the cases Riemann surface is formed by two branches.

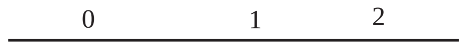


Fig. 52a

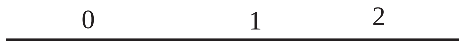


Fig. 52b

At $z = -1$, we note that

$$\operatorname{Arg} z = \operatorname{Arg}(z-1) = \operatorname{Arg}(z-2) = \pi \text{ and } \sqrt{|z(z-1)(z-2)|} = \sqrt{6}.$$

So, f(–1) can be either $\sqrt{6}e^{i3\pi/2} = -\sqrt{6}i$ or $\sqrt{6}e^{i(\pi+2\pi+\pi+2\pi+\pi+2\pi)} = \sqrt{6}e^{i3\pi/2}e^{i\pi} = \sqrt{6}i.$

The branch that gives $f(-1) = \sqrt{6}i$ corresponds to k = 0. With the choice of that branch, we find

$$f(i) = \sqrt{|i(i-1)(i-2)|}\, e^{i|\operatorname{Arg} i + \operatorname{Arg}(i-1) + \operatorname{Arg}(i-2)/2}$$

$$= \sqrt{\sqrt{2\sqrt{5}}}\, e^{i(\pi/2+3\pi/4+\pi-\tan^{-1}1/2)/2} = \sqrt[4]{10}\, e^{i\left(\frac{\pi}{4}-\tan^{-1}\frac{1}{2}\right)/2^{2}} e^{i\pi}$$

$$= -\sqrt[4]{10}\, e^{1(\tan^{-1}1-\tan^{-1}1/2)/2} = -\sqrt[4]{10}\, e^{i(\tan^{-1}1/3)/2}$$

4.11 The Inverse Trigonometric Functions

(i) The function $\sin^{-1}z$ is defined by the equation

$$z = \sin \omega$$

Substituting $\dfrac{e^{i\omega} - e^{-i\omega}}{2i}$ for $\sin \omega$, we find that

$$(e^{i\omega})^2 - 2ie^{i\omega}z - 1 = 0$$

i.e., $e^{i\omega} = iz + (1 - z^2)^{1/2}$

$\Rightarrow$ $i\omega = \log\{iz + (1 - z^2)^{1/2}\}$

so that $\sin^{-1}z = -i\log\{iz + (1 - z^2)^{1/2}\}$

Similarly, we can have

$$\cos^{-1}z = -i\log\{z + (z^2 - 1)^{1/2}\}$$

(ii) We take the function $\omega = \tan^{-1}z$, which is the inverse of $z = \tan\omega$. Expressing $\tan\omega$ in terms of $\sin\omega$ and $\cos\omega$ and then converting to their exponential form, we get

$$z = \frac{1}{i}\frac{e^{i\omega} - e^{-i\omega}}{e^{i\omega} + e^{-i\omega}}$$

$$= \frac{1}{i}\frac{e^{2i\omega} - 1}{e^{2i\omega} + 1}$$

i.e.,
$$iz = \frac{e^{2i\omega} - 1}{e^{2i\omega} + 1} \Rightarrow e^{2i\omega} = \frac{1 + iz}{1 - iz}$$

and finally,
$$\omega = \frac{1}{2i}\log\frac{1 + iz}{1 - iz}$$

Note : When $z \neq \pm 1$, the quantity $(1 - z^2)^{1/2}$ has two possible values. For each value, the logarithm generates infinitely many values. Therefore $\sin^{-1}z$ has two sets of infinite values. For example, consider

$$\sin^{-1}\frac{1}{2} = \frac{1}{i}\log\left(\frac{i}{2} \pm \frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{i}\left[\log e^{i\left(\frac{\pi}{6} + 2k\pi\right)}\right] \text{ or } \frac{1}{i}\left[\log e^{i\left(\frac{5\pi}{6} + 2k\pi\right)}\right]$$

$$= \frac{1}{i}\left[i\left(\frac{\pi}{6} + 2k\pi\right)\right] \text{ or } \frac{1}{i}\left[i\left(\frac{5}{6}\pi + 2k\pi\right)\right]$$

$$= \frac{\pi}{6} + 2k\pi \text{ or } \frac{5\pi}{6} + 2k\pi, \text{ k is any integer.}$$

Likewise, the set of values for other inverse trigonometric functions can be ascertained.

Example 22 : Discuss the mapping $\omega = \sinh z$ that transforms the infinite strip $-\infty < x < \infty$, $0 < y < \pi$ into the ω-plane. Find cuts in the ω-plane which make the mapping continuous both ways. What are the images of the lines (a) $y = 1/\pi$ (b) $x = 1$?

Solution : First we express $\sinh z$ in cartesian form

$$\omega = \sinh z = \sinh x \cos y + i\cosh x \sin y = u + iv$$

We consider the line segment $x = c$, $y \,\varepsilon\, (0, \pi)$. Its image is

$\{\sinh c \cos y + i \cosh c \sin y | y \in (0, \pi)\}$

Clearly, u and v then satisfy the equation for the ellipse

$$\frac{u^2}{\sinh^2 c} + \frac{v^2}{\cosh^2 c} = 1$$

The ellipse starts at the point (sinh c, 0), passes through the point (0, cosh c) and ends at (–sinh c, 0). As c varies from zero to ∞ or from zero to –∞, the semi-ellipses cover the upper-half of ω-plane. Thus the mapping is 2–to–1.

Now consider the infinite line $y = c$, $x \in (-\infty, \infty)$.

It's image is $\{\sinh x \cos c + i \cosh x \sin c | x \in (-\infty, \infty)\}$.

Here u and v satisfy the equation for a hyperbola

$$\frac{u^2}{\cos^2 c} - \frac{v^2}{\sin^2 c} = 1$$

As c varies from 0 to π/2 or from π/2 to π, the semi-hyperbola cover the upper-half of ω-plane. Thus the mapping is 2-to-1.

We look for branch points of $\sinh^{-1}\omega$

$$\omega = \sinh z$$

$$\omega = \frac{e^z - e^{-z}}{2}$$

$$e^{2z} - 2\omega e^z - 1 = 0$$

$$e^z = \omega + (\omega^2 + 1)^{1/2}$$

$$z = \log(\omega + (\omega - i)^{1/2} (\omega + i)^{1/2})$$

The branch points are at $\omega = \pm i$. Since $\omega + (\omega^2 + 1)^{1/2}$ is non zero and finite in the finite complex plane, the logarithm does not introduce any branch in the finite plane. Thus the only branch point in the upper-half of ω-plane is at ω = i. Any branch cut that connects ω = i with the boundary of Im ω > 0 will separate the branches under the inverse mapping.

We consider the line y = π/4. The image under the mapping is the upper-half of the hyperbola

$$2u^2 - 2v^2 = 1$$

Consider the segment x = 1. The image under the mapping is the upper-half of the ellipse.

$$\frac{u^2}{\sinh^2 1} + \frac{v^2}{\cosh^2 1} = 1$$

Example 23 : Construct a Riemann Surface for $\cos^{-1}z$.

Solution : The function $\omega = \cos^{-1}z = -i \log [z + (z^2 - 1)^{1/2}]$ has two sources of multi-valuedness; one due to the square root function $(z^2 - 1)^{1/2}$ and the other due to the logarithm, if any.

First we construct the branch of the square root

$$(z^2 - 1)^{1/2} = (z + 1)^{1/2}(z - 1)^{1/2}$$

We see that there are branch points at $z = -1$ and $z = 1$. In particular we want the $\cos^{-1}z$ to be defined for $z = x$, $x \in [-1, 1]$. Hence we introduce the branch cuts on the lines $(-\infty, -1]$ and $[1, \infty)$. Let

$$z + 1 = re^{i\theta},\ z - 1 = \rho\, e^{i\phi}$$

With the given branch cuts, the angles have the possible ranges

$$-\pi \le \theta < \pi,\ 0 \le \phi < 2\pi$$

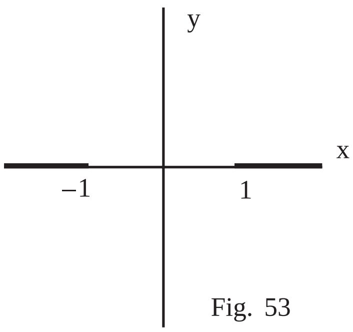


Fig. 53

Now we must determine if the logarithm introduces additional branch points. The only possibilities for branch points are where the argument of the logarithm is zero.

$$z + (z^2 - 1)^{1/2} = 0$$

or, $z^2 = z^2 - 1 \Rightarrow 0 = -1$

We see that the argument of the logarithm can not be zero and thus there are no additional branch points. Here the Riemann surface consists of two sheets S_0 and S_1 joined on the real axis $(-\infty, -1] \cup [1, \infty)$:

$$S_0 \begin{cases} 0 \le \phi < 2\pi \\ -\pi \le \theta < \pi \end{cases} \qquad S_1 \begin{cases} 2\pi \le \phi < 4\pi \\ \pi \le \theta < 3\pi \end{cases}$$

A particular branch of this function can be obtained by first taking

$$z + 1 = re^{i\theta},\ -\pi \le \theta < \pi;\ z - 1 = \rho e^{i\phi},\ 0 \le \phi < 2\pi$$

Then adding these relations, we find

$$z = (re^{i\theta} + \rho e^{i\phi})/2$$

and the function $z + (z^2 - 1)^{1/2}$ reduces to

$$z + (z^2 - 1)^{1/2} = \frac{re^{i\theta} + \rho e^{i\phi}}{2} + (r\rho)^{1/2} e^{i(\theta+\phi)/2}$$

$$= \frac{re^{i\theta}}{2}\left(1 + \frac{\rho}{r} e^{i(\phi-\theta)} + 2\sqrt{\frac{\rho}{r}} e^{i(\phi-\theta)/2}\right)$$

$$= \frac{re^{i\theta}}{2}\left(1+\sqrt{\frac{\rho}{r}}e^{i(\phi-\theta)/2}\right)^2$$

Then $\cos^{-1} z = -i\left\{\log\left(\frac{r}{2}e^{i\theta}\right)+\log\left(1+\sqrt{\frac{\rho}{r}}e^{i(\phi-\theta)/2}\right)^2\right\}$ on S_0. If a point lying on the sheet S_0 is allowed to travel a path making a complete round about only the branch point z = 1, it enters to the sheet S_1 from the sheet S_0. In this case the value of ϕ changes by 2π while the value of θ remains unchanged. The change in $(\phi-\theta)/2$ is π. So in this case,

$\cos^{-1} z = -i\left\{\log\left(\frac{r}{2}e^{i\theta}\right)+\log\left(1-\sqrt{\frac{\rho}{r}}e^{i(\phi-\theta)/2}\right)^2\right\}$ on S_1. Similarly we can analyse the case when the point on S_0 encloses only the branch point z = –1 while travelling a complete round.

Some standard branch cuts of elementary functions.

Function	Branch cuts
z^s, non integral s with Re s > 0	$(-\infty, 0)$
z^s, non integral s with Re s ≤ 0	$(-\infty, 0]$
e^z	none
log z	$(-\infty, 0]$
$\sin^{-1}z$, $\cos^{-1}z$	$(-\infty, -1]$ and $[1, \infty)$
$\tan^{-1}z$	$y \le -1, x = 0$ and $y \ge 1, x = 0$
$\text{cosec}^{-1}z$, $\sec^{-1}z$	$(-1, 1)$
$\cot^{-1}z$	$[-i, i]$
$\sinh^{-1}z$	$y < -1, x = 0$ and $y > 1, x = 0$
$\cosh^{-1}z$	$(-\infty, 1)$
$\text{cosech}^{-1}z$	$-1 < y < 1, x = 0$
$\text{sech}^{-1}z$	$(-\infty, 0]$ and $(1, \infty)$
$\tanh^{-1}z$	$y \le 1, x = 0$ and $y \ge 1, x = 0$
$\coth^{-1}z$	$[-1, 1]$

Exercises

1. Find the principal value of each of the following complex quantities :

   (a) $(1 - i)^{1+i}$ (b) 3^{3-i} (c) 2^{2i}

2. Give the number of branches and locations of the branch points for the functions.

   (a) $\cos (z^{1/2})$ (b) $(z + i)^{-z}$

3. Determine the branch points of the function

   $\omega = \{(z^2 - z)(z + 2)\}^{1/3}$

4. Find the branch points of $(z^{1/2} - 1)^{1/2}$ in the finite complex plane. Introduce branch cuts to make the function single-valued.

5. Let D be the complex z-plane with a cut along the segment [–1, 1], determine the regular branches of the function

$$f(z) = \left(\frac{1-z}{1+z}\right)^{1/2}$$

6. Split the function $f(z) = \sqrt{(z^2-4)(z^2-9)}$ into two regular branches in the domain $D : \mathbb{C} \setminus \{[-3, -2], [2, 3]\}$

7. Evaluate

   (i) $\int_0^\infty \frac{x^\alpha}{x^2 - 1} dx, \quad -1 < \alpha < 1$ (ii) $\int_0^\infty \frac{\log x}{x^2+1} dx$

8. Prove that $\int_0^\pi \log \sin x dx = -\pi \log 2.$

9. Construct a Riemann surface for the following functions :

   (i) $\omega = z^{1/3}$ (ii) $\omega = (z^2 + 1)^{1/2}$ (iii) $\omega = \log \frac{z+1}{z-1}$ (iv) $\omega = \sin^{-1} z.$

10. Let f(z) have branch points at z = 0 and z = ± i but nowhere else in the extended complex plane. How does the value and argument of f(z) change while traversing the contour given in the figures 51(a) (b). Do the branch cuts make the function single valued?

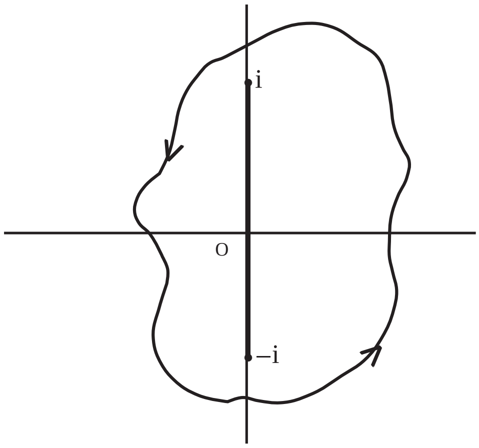


Fig. 54 (a)

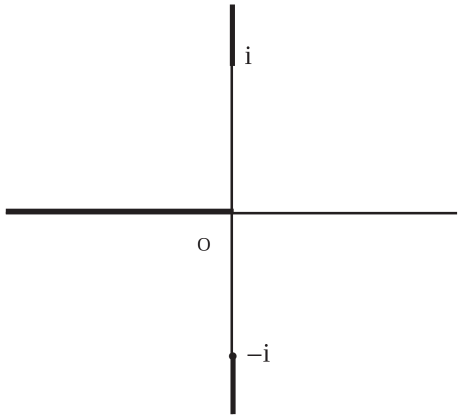


Fig. 54 (b)

Unit 5 ❑ Conformal Equivalence

Structure

5.0 Objectives of this Chapter

The concept of conformal equivalence of two regions will be introduced in this chapter. The main theorem of this chapter is Riemann mapping theorem. Also Hurwitz's theorem, Schwarz lemma, Schwarz reflection principle, Schwarz-Christoffel transformation will be studied and their applications will be shown through a few examples.

5.1 Riemann Mapping Theorem

In the family of analytic functions that concern geometrical orientation, conformal mapping plays a leading role. As its consequences we shall present here a most important result named after G. F. B Riemann, known as "Riemann mapping theorem". Throughout H(G) will denote the family of analytic functions defined on the region G.

Definition : Conformal Equivalence :

Two regions R_1 and R_2 are said to be conformally equivalent if there exists a $f \in H(R_1)$ such that f is one-to-one in R_1 and $f(R_1) = R_2$ i.e. if there exists a conformal mapping one to one of R_1 onto R_2. Clearly, this is an equivalence relation (reflexive, symmetric and transitive).

Theorem 5.1 [Hurwitz's Theorem] Let G be a region and $\{f_n\}$ be a sequence in H(G) that converges uniformly to $f \in H(G)$. Suppose $f \neq 0$, $\overline{D}(a, R) \subset G$ and f(z) $\neq 0$ on $\gamma : |z-a| = R$. Then there exists an integer N such that for $n \geq N$, f_n and f have the same number of zeros in D(a, R).

Proof. Since f(z) is never zero on the circle γ, we have

$$\operatorname*{Inf}_{\gamma}|f(z)| = \delta > 0$$

Again, $f_n \to f$ uniformly on γ, so there is an integer N such that for $n \geq N$

$$\sup_{\gamma}|f_n(z) - f(z)| < \frac{\delta}{2}$$

and thus on the circle γ, $|f(z) - f_n(z)| < \frac{\delta}{2} < \delta \leq |f(z)|$ for $n \geq N$. Using Rouche's theorem we find that f_n and f have the same number of zeros inside the circle $\gamma : |z-a| = R$ for $n \geq N$.

By means of the above theorem, we can easily prove

Corollary 1. Let G be a region and $\{f_n\}$ be a sequence in H(G) such that each f_n never vanishes in G. Suppose $f_n \to f$ uniformly in H(G). Then f(z) never vanishes in G, unless $f \equiv 0$.

Some useful results

(i) If f(z) is analytic at z_0 and $f^1(z_0) \neq 0$, then there is a neighbourhood of z_0 in which f(z) is univalent.

(ii) An univalent analytic function f on a domain G has a non-zero derivative at every point of G, i.e., $f^1(z) \neq 0$ on G.

(iii) The inverse of an univalent analytic function is analytic.

(iv) Any domain in $\mathbb{C}$, that is conformally equivalent to a simply connected domain must itself be simply connected.

(v) A domain D in $\mathbb{C}$ is simply connected if and only if every analytic function in D has a primitive in D.

Schwarz Lemma

Let $f : D(0, 1) \to D(0, 1)$ be an analytic function which maps the unit disc D(0, 1) to itself. If f(0) = 0,

then

(i) $|f(z)| \leq |z|$ for $0 \leq |z| < 1$

(ii) $|f^1(0)| \leq 1$

(iii) if equality holds in (i) for at least one $z \in D(0, 1) - \{0\}$, or, if equality holds in (ii), then

$$f(z) = \lambda z,$$

where λ is a constant, $|\lambda| = 1$.

Proof : Let us consider the function

$$g(z) = \frac{f(z)}{z}$$

which is analytic in the disc D(0, 1) –{0} and it has removable singularity at z = 0, since f(0) = 0. It can be made analytic at z = 0 if we define

$$g(0) = \lim_{z\to 0} \frac{f(z)}{z} = f^1(0) \tag{55}$$

For |z| = r, where 0 < r < 1

$$|g(z)| = \frac{|f(z)|}{|z|} < \frac{1}{r}$$

By the Maximum Modulus Principle, |g(z)| < $^1/_r$ for the entire disc |z| ≤ r. We fix z∈D (0, 1) –{0} and let r → 1. Then

$$|g\ (z)| \le 1.$$

This is true for all z∈D (0,1) –{0} and we get

$$\frac{|f(z)|}{|z|} \le 1,\ 0 < |z| < 1 \tag{56}$$

i.e. |f(z)| ≤ |z|, 0 < |z| < 1. Since f(0) = 0, we have |f(z)| ≤ |z| for 0 ≤ |z| < 1. So,

(i) is proved and we find from (55) that |g (0)| = $|f^1(0)|$ ≤ 1 which proves (ii) To prove (iii), we observe that if at a point $z_0 \neq 0$ ($|z_0| < 1$) $|g(z_0)|$ 1 = 1 i.e. |g(z)| attains its maximum at an internal point and hence by the maximum modulus principle g(z) = λ, a constant and that |λ| = 1, so f(z) = λz.

Theorem 5.2 Let a∈D (0, 1). Then ϕ_a defined by

$$\phi_a(z) = \frac{z - a}{1 - \bar{a}z}$$

maps $\overline{D}$ (0, 1) onto $\overline{D}$ (0, 1).

Proof. Clearly, ϕ_a is a bilinear transformation, it is analytic in the whole complex plane except the point $\frac{1}{\bar{a}}$ (which is the inverse point of the point a with respect to the circle |z| = 1, and hence lies outside |z| = 1). We observe that

$$\phi_a(\phi_{-a}(z)) = \frac{\dfrac{z+a}{1+\bar{a}z} - a}{1 - \bar{a}\dfrac{z+a}{1+\bar{a}z}}$$

$$= \frac{z\left(1 - |a|^2\right)}{1 - |a|^2}$$

$= z = \phi_{-a}(f_a(z))$, similarly.

Thus ϕ_a maps D (0, 1) onto D (0, 1) in a one to one way. Now let θ be a real number. Then

$$\left|\phi_a\left(e^{i\theta}\right)\right| = \left|\frac{e^{i\theta} - a}{1 - \bar{a}e^{i\theta}}\right|$$

$$= \left|\frac{e^{i\theta} - a}{e^{-i\theta} - \bar{a}}\right| \left|\frac{1}{e^{i\theta}}\right| = \left|\frac{e^{i\theta} - a}{\overline{e^{i\theta} - a}}\right| = 1$$

i.e., ϕ_a maps $|z| = 1$ on $|z| = 1$. Thus, ϕ_a maps $\overline{D}$ (0, 1) onto $\overline{D}$ (0, 1).

A maximal problem

Let α, β be two complex numbers with $|\alpha| < 1$, $|\beta| < 1$ and f be analytic on D(0, 1) satisfying $f(\alpha) = \beta$. What is the maximum possible value of $|f^1(\alpha)|$ among such mappings?

Solution : Let

$$g = \phi_\beta \; 0 \; f \; 0 \; \phi_{-\alpha} \text{ where } \phi_\beta \text{ is defined as in theorem 5.2} \qquad (57)$$

Then g maps D (0, 1) to D (0, 1) and satisfies

$$\begin{aligned} g(0) &= \phi_\beta\{f(\phi_{-\alpha}(0))\} \\ &= \phi_\beta\{f(\alpha)\} \\ &= \phi_\beta(\beta) \\ &= 0 \end{aligned}$$

Thus g satisfies all the conditions of Schwaz's lemma and hence $|g^1(0)| \le 1$. To obtain an explicit form of $g^1(0)$, we use (57) and apply the chain rule

$$\begin{aligned} g^1(0) &= \{(\phi_\beta 0 f)^1 (\phi_{-\alpha}(0))\}\phi^1_{-\alpha}(0) \\ &= (\phi_\beta 0 f)^1 (\alpha)(1-|\alpha|^2) \\ &= \phi_\beta^1(f(\alpha))f^1(\alpha)(1-|\alpha|^2) \\ &= \phi_\beta^1(\beta)f^1(\alpha)(1-|\alpha|^2) \\ &= \frac{1-|\alpha|^2}{1-|\beta|^2} f^1(\alpha) \end{aligned}$$

But $|g^1(0)| \le 1$, therefore

$$\left|f^1(\alpha)\right| \le \frac{1-|\beta|^2}{1-|\alpha|^2} \qquad (58)$$

Equality in (58) occurs only when $|g^1(0)| = 1$. In that case by virtue of Schwarz lemma there is a constant λ, $|\lambda| = 1$ so that $g(z) = \lambda z$. Hence,

$$f(z) = \phi_{-\beta}\{\lambda\phi_{\alpha}(z)\}, \qquad z \in D(0, 1) \tag{59}$$

We now present an important consequence of Schwarz's lemma, which may be seen as the converse form of theorem 5.2.

Theorem 5.3 : Let $f : D(0, 1) \to D(0, 1)$ be any conformal map of the unit disc onto itself and $f(a) = 0$, $a \in D(0, 1)$. Then there is a constant λ, $|\lambda| = 1$ such that

$f(z) = \lambda\phi_a(z)$ where ϕ_a is defined as in theorem 5.2.

Proof. Since f is a conformal map from D(0, 1) to D (0, 1), we can have inverse of f, g defined by

$$g\{f(z))\} = z,$$

which is analytic too. Applying the chain rule

$$g^1(0)\, f^1(a) = 1 \tag{60}$$

But according to inequality (58), f and g have to satisfy

$$\left|f^1(a)\right| \le \frac{1}{1-|a|^2}, \qquad \left|g^1(0)\right| \le 1-|a|^2 \tag{61}$$

(since, $f(a) = 0$ and $g(0) = a$).

From (60), (61) it follows that $|f^1(a)| = (1 - |a|^2)^{-1}$. Hence applying the result (59) we find that

$$f(z) = \lambda\phi_a(z)$$

for some λ with $|\lambda| = 1$.

Lemma 5.1 : Let G be a simply connected region and $\{f_n\}$ be a sequence of injective analytic mappings (conformal mappings) of G into $\mathbb{C}$ which converges uniformly on every compact subset of G, then the limit function f is either constant or injective.

Proof. Suppose f is not constant and not injective. Then there exist two points ς and $\eta \in G$, $\varsigma \neq \eta$ such that $f(\varsigma) = f(\eta) = \omega_0$, say.

Let $g_n(z) = f_n(z) - \omega_0$. We can find a positive δ, $\delta < |\varsigma - \eta|/2$ so that the discs $D(\varsigma, \delta)$ and $D(\eta, \delta)$ are included in G. Now $g(z) = f(z) - \omega_0$ never vanishes on the circles $|z - \varsigma| = \delta$ and $|z - \eta| = \delta$, where $g(z) = \lim_{n\to\infty} g_n(z)$. Applying Hurwitz's theorem, for large n, there exists ς_n lying inside the circle $|z - \varsigma| = \delta$ with $g_n(\varsigma_n) = 0$ as $g_n \to g$ uniformly in G. Similarly, for all large n, there is η_n within $|z - \eta| = \delta$ with $g_n(\eta_n) = 0$. But by construction, $D(\varsigma, \delta) \cap D(\eta, \delta) = \phi$ and hence $\varsigma_n \neq \eta_n$. Thus

$$g_n(\varsigma_n) = g_n(\eta_n) = 0, \ \varsigma_n \neq \eta_n$$

that is, $$f_n(\varsigma_n) = f_n(\eta_n), \ \varsigma_n \neq \eta_n$$

contradicting the injectivity of each f_n and the proof follows.

NOTE : There is no conformal map f of the unit disc D (0, 1) onto the whole complex plane $\mathbb{C}$ because then the inverse function $f^{-1} : \mathbb{C} \to D(0, 1)$ would be a bounded entire function which is not constant, contradicting the Liouville's theorem.

Open mapping theorem : Let G be a region and suppose that f is a non-constant analytic function on G. Then for any open set U in G, f(U) is open.

Proof : Omitted.

Uniform boundedness : A sequence of functions $\{f_n\}$ defined on a set D is said to be uniformly bounded on D if there exists a constant $M > 0$ such that $|f_n(z)| \leq M$ for all n and for all $z \in D$.

Normal family : Let F be a family of functions in a region G. The family F is said to be normal in G if every sequence $\{f_n\}$ of functions $f_n \in F$ contains a subsequence $\{f_{n_k}\}$ which converges uniformly on every compact subset of G.

Montel's theorem : A family F in H (G) is normal if and only if F is uniformly bounded on every compact subset of G.

Proof : Omitted.

Theorem 5.4 : [Riemann Mapping Theorem] Let G be a simply connected region, except for $\mathbb{C}$ itself and let $a \in G$. Then there is a unique conformal map $f : G \to D(0, 1)$ of G onto the unit disc which satisfies

$$f(a) = 0 \text{ and } f^1(a) > 0.$$

Proof. Let us first prove that f is unique. If there was another conformal map $g : G \to D(0, 1)$ with the given properties, then

$$fog^{-1} : D(0, 1) \to D(0, 1)$$

would be a conformal map and also

$$(fog^{-1})(0) = f(a) = 0$$

Hence, applying Theorem 5.3, we find that there is a constant λ with $|\lambda| = 1$

$$(fog^{-1})(z) = \lambda z$$

Deriving the derivative at the origin, we find

$$(fog^{-1})'(0) = f'(g^{-1})(0)(g^{-1})'(0) = f'(a)\frac{1}{g'\left(g^{-1}(0)\right)} = \frac{f'(a)}{g'(a)} > 0,$$

from which it follows that λ is positive. But also $|\lambda| = 1$, so $\lambda = 1$. Thus fog^{-1} is an identity map and f = g.

The proof of existence is divided into several stages.

Lemma 5.2 Let G be a simply connected region other than $\mathbb{C}$. Then there exists an injective analytic map f on G with $f(G) \subset D(0, 1)$.

Proof. We choose a point $b \in \mathbb{C} \backslash G$. Since G is simply connected there exists a $g : G \to \mathbb{C}$ analytic with $g^2(z) = z - b$.

Here g is injective since

$$g(z_1) = g(z_2)$$

$$\Rightarrow \quad g^2(z_1) = g^2(z_2)$$

$$\text{i.e.} \quad z_1 - b = z_2 - b$$

$$\Rightarrow \quad z_1 = z_2.$$

By open mapping theorem g(G) is open. Let us pick $\omega_0 \in g(G)$ and choose $r > 0$ so that $D(\omega_0, r) \subset g(G)$. Then $D(-\omega_0, r) \subset \mathbb{C} \setminus g(G)$. For, if there exists a point $\omega \in D(-\omega_0, r) \cap g(G)$, then $\omega = g(z_1)$ for some $z_1 \in G$ and also $-\omega \in D(\omega_0, r) \subset g(G)$, so that $-\omega = g(z_2)$ for some $z_2 \in G$. Again,

$$g(z_1) = -g(z_2)$$

$$\Rightarrow \quad g^2(z_1) = g^2(z_2)$$

$$\text{or,} \quad z_1 - b = z_2 - b$$

$$\text{i.e.} \quad z_1 = z_2$$

$$\text{or,} \quad g(z_1) = g(z_2) = -g(z_1)$$

$$\Rightarrow \quad g(z_1) = 0$$

$$\Rightarrow \quad 0 = g^2(z_1) = z_1 - b$$

$$\text{i.e.} \quad z_1 = b \in \mathbb{C} \setminus G$$

contradicting $z_1 \in G$.

We take
$$f(z) = \frac{r}{2[g(z) + \omega_0]} \tag{62}$$

Then f is injective analytic map on G (by construction $|g(z) + \omega_0| \geq r$ for $z \in G$) and also satisfies $|f(z)| \leq \frac{1}{2} < 1$ for $z \in G$.

Lemma 5.3 : Let G be a simply connected region other than $\mathbb{C}$ itself and let $a \in G$ be fixed. Then there exists a conformal map $f : G \to D(0, 1)$ of G onto the unit disc with the properties $f(z) = 0$ and $f(a) > 0$.

Proof : Let F denote the family of analytic functions $f : G \to \mathbb{C}$ such that either $f \equiv 0$ or f is injective, and $f(G) \subset (0, 1)$, $f(a) = 0$ and $f'(a) > 0$.

Let us consider the function

$$\psi(z) = \frac{f(z) - f(a)}{1 - \overline{f(a)}\, f(z)}$$

where f(z) is given by (62) of lemma 5.2 and we find that $\psi(G) \subset D(0, 1)$, $\psi(a) = 0$ and $\psi^1(a) > 0$. So F is non empty and by Montel's theorem it is normal. Applying Lemma 1 we see that all functions in the closure of F in H(G) are either constant or injective. Now since all functions in F take the value zero at a, the same is true for all functions in the closure of F. Likewise the only constant function in the closure is

0 while the other functions in the closure satisfy f(G) ⊂ $\overline{D}$ (0, 1). Since f(G) is open, by open mapping theorem, f(G) ⊂ D (0, 1). Again since the f → $f^1(a)$ is continuous, all functions in the closure of F must satisfy $f^1(a) \geq 0$. The functions in the closure, that are not identically zero, are injective, so $f^1(a) > 0$ unless f ≡ 0. These observations prove that the set F is closed in H(G). Hence F is compact in H(G).

Since the map f → f′(a) : F → R is a continuous function on a compact set, it must attain its maximum value, as we are not considering constant function (here it is zero). Let f∈F be a function with f′(a) maximum.

We now show that f(G) = D (0, 1). On the contrary, suppose that f(G) ≠ D (0, 1) and choose w∈ D(0, 1)\f(G). Using the property that every non-vanishing analytic function in a simply connected region has an analytic square root, we take a function h ∈H(G) with

$$[h(z)]^2 = \frac{f(z)-\omega}{1-\overline{\omega}f(z)} \tag{63}$$

Now as the bilinear transformation $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$ maps D (0, 1) onto D (0, 1) and as f∈F, h(G) ⊂ D (0, 1).

Let g : G → $\mathbb{C}$ defined by

$$g(z) = \frac{|h'(a)|}{h'(a)} \cdot \frac{h(z)-h(a)}{1-\overline{h(a)}\,h(z)}$$

Then clearly, g(G) ⊂ D (0, 1), g(a) = 0 and g is analytic injective and g′(a) > 0, since

$$g'(a) = \frac{|h^1(a)|}{h^1(a)} \cdot \frac{h^1(a)[1-|h(a)|^2]}{[1-|h(a)|^2]^2}$$

$$= \frac{|h^1(a)|}{1-|h(a)|^2} > 0 \tag{64}$$

So, g∈F.

Again, differentialing (63) we find that

$$2h(a)h^1(a) = f^1(a)(1-|\omega|^2)$$

So, from (64)

$$g^1(a) = \frac{|h(a)||h^1(a)|}{|h(a)|(1-|h(a)|^2} = \frac{f^1(a)(1-|\omega|^2}{2\sqrt{\omega}(1-|\omega|)}, \text{ as } |h(a)|^2 = |\omega|$$

$$= \frac{f^1(a)\left(1+|\omega|\right)}{2\sqrt{\omega}} > f^1(a).$$

contradicting the choice of $f \in F$ as maximising $f^1(a)$. Thus f(G) = D (0, 1).

Note : The Riemann mapping theorem is one of the most celebrated results of complex analysis. It is the beginning of the study of complex analysis from a geometric view point. G. F. B. Riemann in 1851 correctly formulated the theorem, but unfortunately his proof of the theorem was lacking. According to various accounts, he assumed but did not prove that a certain maximal problem had a solution. A final proof was definitely known by the early 20th century, different sources attributed to it particularly, W. F. Osgood, P. Koebe, L Bieberbach etc.

5.2 The Schwarz Reflection Principle

Let f be analytic in the domains D_1, D_2 which have a common piece of boundary, a smooth curve γ. Assume further that f is continuous across γ. Then, by Morera's theorem, f is analytic in $D_1 \cup D_2$. This allows us to perform analytic continuation in some cases.

Theorem 5.5 [The Schwarz reflection principle] Given a function f(z) analytic in a domain D lying in the upper half plane whose boundary contains a segment $I \subset$ IR, assume f is continuous on $D \cup I$ and real-valued on I. Then f has analytic continuation across I, in a domain $D \cup I$ U D*, where D* $= \{\bar{z} : z \in D\}$.

Proof. Let us consider the function

$$f(z) = \begin{cases} f(z), & z \in DUI \\ \overline{f(\bar{z})}, & z \in D^* UI \end{cases}$$

It is clear that F is analytic in D. We shall show that F is also analytic in D*. Let z and z + h lie within D*. Then $\bar{z}$ and $\bar{z}+\bar{h}$ lie within D and we can express.

$$\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = \lim_{h\to 0} \frac{\overline{f(\bar{z}+\bar{h})}-\overline{f(\bar{z})}}{h} = \lim_{h\to 0} \left[\frac{\overline{f(\bar{z}+\bar{h})-f(\bar{z})}}{h}\right] = \overline{f'(\bar{z})}.$$

So, F is analytic in D*. F is also continuous on D*U I.

For, z $\in$ I

$$\lim_{z\to x} F(z) = \lim_{z\to x} \overline{f(\bar{z})} = \overline{f(x)} = f(x),$$

by hypothesis. Thus F is continuous on D U I U D*. To prove F is also analytic there, we consider the function

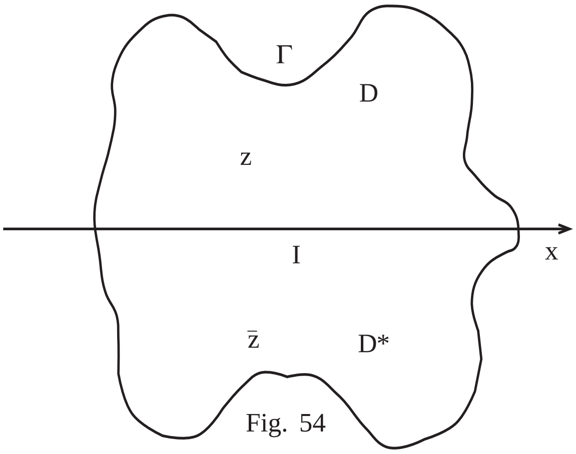


Fig. 54

$$\phi(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{F(\varsigma)}{\varsigma - z}\, d\varsigma \qquad (65)$$

It is analytic in D U I U D* [as (i) $\frac{F(\varsigma)}{\varsigma - z}$ is continuous function of both variables when z lies within Γ and ς on Γ.

(ii) for each such ς, $\frac{F(\varsigma)}{\varsigma - z}$ is analytic in z in D U I U D*. [see (14)].

To complete the proof, we try to establish φ(z) = F(z) for all z ε D U I D*.

Breaking the integral in (65) and adding the two integrals along I, which are in opposite directions, we write

$$\phi(z) = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{F(\varsigma)}{\varsigma - z}\, d\varsigma + \frac{1}{2\pi i}\int_{\Gamma_2} \frac{F(\varsigma)}{\varsigma - z}\, d\varsigma \qquad (66)$$

where Γ_1 and Γ_2 are the boundary of D U I and D* U I respectively. When z∈ DUI, the second integral in (66) vanishes and φ(z) = F(z). Again, the first integral vanishes when z ε D* U I and φ(z) = F(z) in this case too. Thus φ(z) = F(z) for all z∈ D U I U D* and we have found a function F(z), analytic in D U I U D*, and coincides with f(z) in D U I.

5.3 The Schwarz-Christoffel Transformation

We know from Riemann's mapping theorem that there is a conformal mapping which maps a given simply connected domain onto another simply connected domain, or equivalently onto the unit disc. But it does not help us to determine such mappings.

Many applications in boundary-value problem requires construction of one-to-one conformal mapping from the upper half plane Im z > 0 onto a polygon Ω in the w-plane. Two German mathematicians H. A. Schwarz and E. B. Christoffel independently discovered a method for finding such mappings during the years 1864-1869.

Theorem 5.6 [Schwarz and Christoffel] Let P be a polygon with vertices w_1, ...w_k in the anticlockwise direction and interior angles $\alpha_1\pi$, ..., $\alpha_k\pi$ respectively, where $-1 < \alpha_1, ..., \alpha_k < 1$. Then there exists a one-to-one conformal mapping of the form

$$f(z) = A\int_{z_0}^{z} (s - x_1)^{\alpha_1 - 1}(s - x_2)^{\alpha_2 - 1} \ldots (s - x_{k-1})\alpha_{k-1} - 1\, ds + B \qquad (67)$$

where A, B$\in \mathbb{C}$, that maps the upper plane Im z > 0 onto the interior of P, with

$$f(x_1) = w_1 ...., f(x_{k-1}) = w_{k-1}, f(\infty) = w_k. \quad (68)$$

Remarks : (i) We do not need to have specific information on w_k and α_k. While travelling the polygon anticlockwise direction we made a left turn of an angle $\pi - \alpha_j \pi$ at the vertex ω_j.

(ii) Sometimes certain infinite regions can be thought of as infinite polygons. In this case it is convenient to take w_k as the point at infinity, as we need no information on α_k.

(iii) It can be shown that Schwarz-Christoffel transformation can be uniquely determined by three points as in the case of bilinear transformation. One of these is used by taking $f(\infty) = \omega_k$. We can therefore have the freedom to choose two points say, x_1 and x_2 satisfying $-\infty < x_1 < x_2 < \infty$.

(iv) Note that the integral involved may be impossible to calculate theoretically. In practical problems numerical techniques are often used to evaluate the integral. In first part of the proof we take $f(x_k) = \omega_k$, x_k = finite.

Proof. By Riemann mapping theorem such a mapping exists. We shall prove that its form is given by (67). So f(z) is analytic for Im z > 0 and $f^1(z) \neq 0$ in the upper half plane. From these it is clear that

$$\frac{d}{dz}\log f'(z) = \frac{f''(z)}{f'(z)}$$

is analytic in the upper half plane. To construct the function f(z) our aim is to establish that $f''(z)/f'(z)$ is analytic for Im z ≥ 0 save for the pre-image points of the vertices of the polygon lying on the real axis.

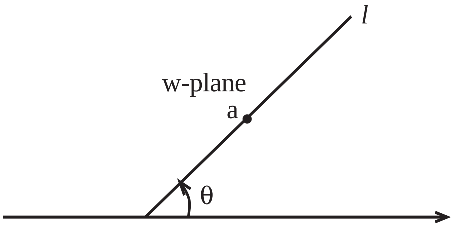


Fig. 55

Let *l* be a side of the polygon P, which makes an angle θ (positive sense) with the real-axis and ς be any point on *l* but not a vertex of the polygon P. Then for any ω on *l*, $(\omega - \varsigma)e^{-i\theta}$ is real and there is a point z on the real axis of the z-plane so that f(z) = ω and a corresponding point z = a for ς on the same line. Hence

$$\{f(z) - \varsigma\}e^{-i\theta}$$

is real and continuous on the segment γ of the real axis of the z-plane corresponding to the straight line *l* of the ω-plane. Moreover, this function is also analytic for Im z > 0, thus following the Schwarz reflection principle we can continue this function analytically across γ to the lower half plane Im z < 0. In particular, this function is analytic in a neighbourhood of the point z = a and can be expanded in the form of the Taylor series.

$$\{f(z)-\varsigma\}e^{-i\theta}=\sum_{k=1}^{\infty}c_k(z-a)^k$$

where $c_1 = f'(a) \neq 0$, maintaining the status quo that $f(a) = \varsigma$ and the function f maps the segment γ onto the straight line *l*. Now

$$f'(z) = e^{i\theta}\{c_1 + c_2 2(z-a) + ...\}$$

and $$\log f'(z) = i\theta + \log\{c_1 + 2c_2(z-a) + ...\}$$

So, $\frac{d}{dz}\log f^1(z)$ is analytic in a neighbourhood of z = a and real on a real line segment intercepted by the neighbourhood.

Let us consider the case when the point ς is the corresponding point at infinity on γ (in this case γ is divided into two parts, each of infinite lenght). Here the Taylor series expansion in the neighbourhood of point at infinity

$$\{f(z)-\varsigma\}e^{-i\theta}=\sum_{k=1}^{\infty}c_k / z^k$$

where each c_R is real and $c_1 \neq 0$ (with the same reason mentioned in the finite case). So

$$f'(z)e^{-i\theta}=-\frac{c_1}{z^2}-\frac{2c_2}{z^3}-\frac{3c_3}{z^4}-...$$

$$f''(z)e^{-i\theta}=\frac{2c_1}{z^3}+\frac{6c_2}{z^4}+\frac{12c_3}{z^5}+.....$$

and we find that

$$\frac{f''(z)}{f'(z)}=\frac{z^{-3}\left\{2c_1+\frac{6c_2}{z}+\frac{12c_3}{z^2}+...\right\}}{-c_1z^{-2}\left\{1+\frac{2c_2/c_1}{z}+...\right\}}=-\frac{1}{c_1}\left\{2c_1+\frac{6c_2}{z}+...\right\}\left\{1-\frac{2c_2/c_1}{z}+...\right\}$$

$$=-\frac{2}{z}+\sum_{k=2}^{\infty}\frac{\tilde{c}_k}{z^k} \tag{69}$$

$\frac{d}{dz}\log f^1(z)$ is analytic in a neighbourhood of the point at infinity and is real when z is real.

In the polygon P, let ℓ^1 be an adjacent side to ℓ making on angle $\alpha_1\pi$ at their point of intersection ω_1. The corresponding point of ω_1 on the real axis is x_1. Here the function f(z) is not analytic in a neighbourhood of x_1, we choose the branch of the argument so that

$$\frac{\pi}{2} < \text{Arg}(z - x_1) < \frac{3\pi}{2}$$

introducing a branch cut along the axis $\{x_1 + iy : y \le 0\}$ [f'(z) is not continuous on this branch cut].

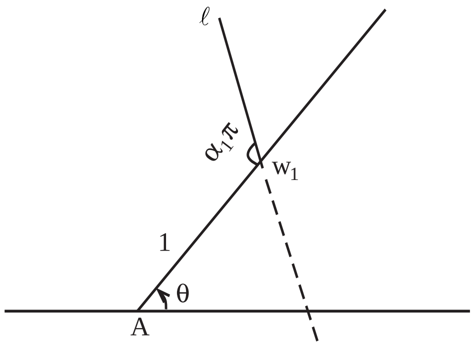


Fig. 56

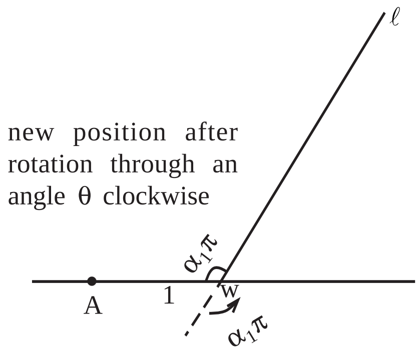


Fig. 57

Here Arg $\{(\omega_1 - \omega)e^{-i\theta}\}$ is equal to zero or $\alpha_1\pi$ according as ω lies on ℓ or ℓ^1. So the function

$$[\{\omega_1 - f(z)\}e^{-i\theta}]^{1/\alpha_1}$$

is real and continuous on the segment of the real axis corresponding to the consecutive sides ℓ and ℓ^1. Again this function is analytic for Im z > 0 since $f(z) - \omega_1$ is analytic and non zero there.

Expanding $\left[\{\omega_1 - f(z)\}e^{-i\theta}\right]^{1/\alpha_1}$ in Taylor's series in a neighbourhood of x_1 we find

$$\left[\{\omega_1 - f(z)\}e^{-i\theta}\right]^{1/\alpha_1} = \sum_{k=1}^{\infty} c_k (z - x_1)^k$$

where each c_k is real and $c_1 \ne 0$. On simplifying, we find

$$f(z) = \omega_1 - e^{i\theta}(z - x_1)^{\alpha_1}[c_1 + c_2(z - x_1) + \dots]^{\alpha_1}$$

$$= \omega_1 + e^{i\theta}(z - x_1)^{\alpha_1} \sum_{k=0}^{\infty} c_k^{\ 1}(z - x_1)^k$$

where c_0^1 is a constant multiple of c_1, hence not equal to zero. Now we have

$$f'(z) = e^{i\theta}(z - x_1)^{\alpha_1 - 1}\left[\alpha_1 c_0^{\ 1} + (\alpha_1 + 1)c_1^{\ 1}(z - x_1) + \dots\right]$$

$$= (z - x_1)^{\alpha_1 - 1}F(z)$$

where F(z) is analytic and not zero in a neighbourhood of z = x_1 and we obtain

$$\frac{d}{dz}\log f^1(z) = \frac{\alpha_1 - 1}{z - x_1} + \frac{F^1(z)}{F(z)} \tag{70}$$

This shows that if the polygon P has an angle $\alpha_1\pi$ at a point ω_1 then $\frac{d}{dz}\log f^1(z)$ will have a simple pole of residue $\alpha_1 - 1$ at its corresponding point x_1.

Now if the point at infinity be the corresponding point to ω_1 at which the polygon P has an angle $\alpha_1\pi$, then we can express

$$\left[\{\omega_1 - f(z)\}e^{-i\theta}\right]^{1/\alpha_1} = \frac{c_1}{z} + \frac{c_2}{z^2} + \ldots.$$

or,
$$f(z) = \omega_1 - e^{i\theta}\left(\frac{c_1}{z}\right)^{\alpha_1}\left(1 + \alpha_1\frac{c_2}{zc_1} + \ldots\right)$$

$$f'(z) = +e^{i\theta}\alpha_1\frac{c_1^{\alpha_1}}{z^{\alpha_1+1}}\left(1 + \alpha_1\frac{c_2}{zc_1} + \ldots\right) - e^{i\theta}\left(\frac{c_1}{z}\right)^{\alpha_1}\left(-\frac{\alpha_1 c_2}{z^2 c_1} - \ldots\right)$$

$$= e^{i\theta}c_1^{\alpha_1}\frac{\alpha_1}{z^{\alpha_1+1}}\left[1 + (\alpha_1 + 1)\frac{c_2}{zc_1} + \ldots\right]$$

$$f''(z) = -e^{i\theta}c_1^{\alpha_1}\frac{\alpha_1(\alpha_1+1)}{z^{\alpha_1+2}}\left\{1 + (\alpha_1+1)\frac{c_2}{zc_1} + \ldots\right\} + e^{i\theta}c_1^{\alpha}\frac{\alpha_1}{z^{\alpha_1+1}}\left\{-(\alpha_1+1)\frac{c_2}{z^2 c_1} - \ldots\right\}$$

$$= -e^{i\theta}c_1^{\alpha_1}\frac{\alpha_1(\alpha_1+1)}{z^{\alpha_1+2}}\left[1 + (\alpha_1+2)\frac{c_2}{zc_1} + \ldots\right]$$

$$\frac{d}{dz}\log f'(z) = \frac{f''(z)}{f'(z)} = -\frac{\alpha_1+1}{z}\left\{1 + (\alpha_1+2)\frac{c_2}{zc_1} + \ldots\right\}\left\{1 - (\alpha_1+1)\frac{c_2}{zc_1} + \ldots.\right\}$$

$$= -\frac{\alpha_1+1}{z}\left\{1 + (\alpha_1 + 2 - \alpha_1 - 1)\frac{c_2}{zc_1} + \ldots.\right\}$$

$$= -\frac{\alpha_1+1}{z} + \sum_{k=2}^{\infty}\frac{\tilde{c}_k}{z^k} \tag{71}$$

Now since $x_2, x_3 \ldots, x_k$ are the corresponding points lying on the real-axis of the z-plane, to the vertices $w_2, w_3, \ldots w_k$ respectively of the polygon P with angles $\alpha_2\pi$,

$\alpha_3\pi$, ... $\alpha_k\pi$ there, the function $\frac{d}{dz}\log f^1(z)$ will have simple poles with residue α_j-1 at x_j, j = 2, ..., k. Thus we see that this function is analytic for Im z > 0 and continuous on Im z = 0 except the points x_1, x_2, ..., x_k and using the Schwarz reflection principle it can be continued analytically across the real axis. Hence $\frac{d}{dz}\log f^1(z)$ possesses only simple poles at x_1, x_2, ... x_k as its only singularities and can be expressed as

$$\frac{d}{dz}\log f^1(z) = \frac{\alpha_1 - 1}{z - x_1} + \frac{\alpha_2 - 1}{z - x_2} + \ldots + \frac{\alpha_k - 1}{z - x_k} + G(z) \tag{72}$$

where G(z) is a polynomial.

When |z| is large enough

$$\frac{\alpha_i - 1}{z - x_i} = \frac{\alpha_i - 1}{z}\left(1 + \frac{x_i}{z} + \frac{{x_i}^2}{z^2} + \ldots\right), i = 1, \ldots, k$$

So, $$\frac{d}{dz}\log f^1(z) = \sum_{i=1}^{k}(\alpha_i - 1)/z + \sum_{i=1}^{k}x_1(\alpha_i - 1)/z^2 + \sum_{i=1}^{k}{x_1}^2(\alpha_1 - 1)/z^3 + \ldots + G(z)$$

$$= -\frac{2}{z} + \sum_{i=2}^{\infty}\frac{d_i}{z_i} + G(z) \tag{73}$$

Using the property of the sum of the exterior angles of a polygon, $(1 - \alpha_1)\pi + (1 - \alpha_2)\pi + \ldots (1 - \alpha_k)\pi = 2\pi$. Comparing (73) with (69) we get G(z) identically zero.

Finally integrating equation (72), we find the desired mapping f(z) as

$$f(z) = A\int_{z0}^{z}(s - x_1)^{\alpha_1 - 1}(s - x_2)^{\alpha_2 - 1}\ldots(s - x_k)^{\alpha_{k-1} - 1}ds + B \tag{74}$$

Role of constants A and B

(i) |A| controls the size of the polygon

(ii) Arg A and B help to select the position, if any, in determining orientation and translation respectively.

An useful observation

In some occasions we urge to make the evaluation process of the integral in (74) simple. For this sake, we consider the point at infinity corresponds to the vertex w_k where the polygon P has an angle $\alpha_k\pi$. Then we can express [see eq. (71)]

$$\frac{d}{dz}\log f^1(z) = \frac{\alpha_k - 1}{z} + \sum_{2}^{\infty}\frac{\tilde{c}_i}{z^i} \tag{75}$$

in the neighbourhood of the point at infinity.

Again considering the expression of $\frac{d}{dz}\log f^1(z)$ in the neighbourhood of the points corresponding to the vertices w_1, w_2 ..., w_{k-1} [see eq. (70)].

$$\frac{d}{dz}\log f^1(z) = \frac{\alpha_1 - 1}{z - x_1} + \frac{\alpha_2 - 1}{z - x_2} + ... + \frac{\alpha_{k-1} - 1}{z - x_{k-1}} + G(z) \qquad (75^1)$$

where G(z) is a polynomial. If |z| is large enough, proceeding as earlier

$$\frac{d}{dz}\log f^1(z) = \sum_1^{k-1}(\alpha_i - 1)/z + \sum_i^{k-1} x_i(\alpha_i - 1)/z^2 + \sum_1^{k-1} x_i{}^2(\alpha_i - 1)/z^3 + G(z)$$

$$= -\frac{\alpha_k + 1}{z} + \sum_2^{\infty}\frac{\tilde{d}_i}{z^1} + G(z) \qquad (76)$$

Comparing (76) with (75), G(z) turns out to be identically zero and hence integrating (75^1) we obtain

$$f(z) = A\int_{z_0}^{z}(s - x_1)^{\alpha_1 - 1}(s - x_2)^{\alpha_2 - 1}...(s - x_{k-1})^{\alpha_{k-1} - 1}ds + B$$

where the role of the constants A and B remain as before.

5.4 Examples : Triangles / Rectangles

The Schwarz-Christoffel transformation is expressed in terms of the points x_j, not in terms of their images i.e., the vertices of the polygon. Not more than three points (x_j) can be chosen arbitrarily. If the point at infinity be one of the x_j's then only two finite points on the real-axis are free to be chosen, whether the polygon is a triangle or a rectangle etc.

Triangle

Let the polygon be a triangle with vertices w_1, w_2 and w_3. The S-C transformation is written as

$$w = A\int_{z_0}^{z}(s - x_1)^{\alpha_1 - 1}(s - x_2)^{\alpha_2 - 1}(s - x_3)^{\alpha_3 - 1}ds + B \qquad (77)$$

where α_1, π, $\alpha_2\pi$ and $\alpha_3\pi$ are the internal angles at the respective vertices.

Fig. 58

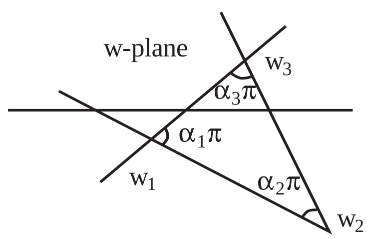


Fig. 59

Here we have chosen all the three finite points x_1, x_2, x_3 on the real-axis.

The constants A, B control the size and position of the triangle respectively.

If we take the vertex w_3 as the image of the point at infinity, the S-C transformation becomes

$$w = A\int_{z_0}^{z} (s - x_1)^{\alpha_1 - 1}(s - x_2)^{\alpha_2 - 1} ds + B \qquad (78)$$

Here x_1 and x_2 can be chosen arbitrarily.

Example 1 : Find a Schwarz-Christoffel transformation that maps the upper half-plane to the inside of the triangle with vertices –1, 1 and √3i.

Solution :

Fig. 60

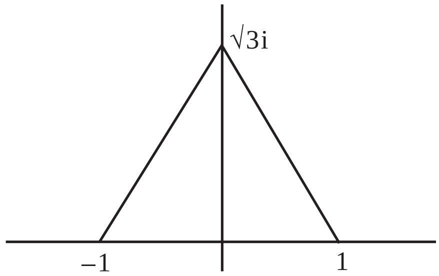


Fig. 61

Following our notation, we write $w_1 = -1$, $w_2 = 1$ and $w_3 = \sqrt{3}i$ so that $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$. We choose the form (78) of S-C transformation and consider the mapping.

$$f(z) = A\int_{0}^{z} (s - x_1)^{-2/3}(s - x_2)^{-2/3} ds + B, \quad [\text{here } f(\infty) = \sqrt{3}i]$$

We may choose $x_1 = -1$ and $x_2 = 1$, so that $f(-1) = -1$ and $f(1) = 1$. Therefore

$$f(z) = A\int_{0}^{z} (s + 1)^{-2/3}(s - 1)^{-2/3} ds + B$$

$$= A\int_{0}^{z} (s^2 - 1)^{-2/3} ds + B$$

It then follows that

$$= A\int_{0}^{-1} (s^2 - 1)^{-2/3} ds + B = -1, \quad A\int_{0}^{1} (s^2 - 1)^{-2/3} ds + B = 1.$$

Rewriting these as

$$-AL + B = -1 \text{ and } AL + B = 1, \text{ where } L = \int_{0}^{1} (s^2 - 1)^{-2/3} ds$$

We obtain $\quad A = \dfrac{1}{\int_{0}^{1} (s^2 - 1)^{-2/3} ds}$ and $B = 0$. Hence

$$f(z) = \frac{1}{\int_0^1 (s^2-1)^{-2/3}ds} \int_0^z (s^2-1)^{-2/3} ds.$$

Example 2 : Using Schwarz-Christoffel transformation map the upper half-plane onto an equilateral triangle of side 5 units.

Solution :

Fig. 62

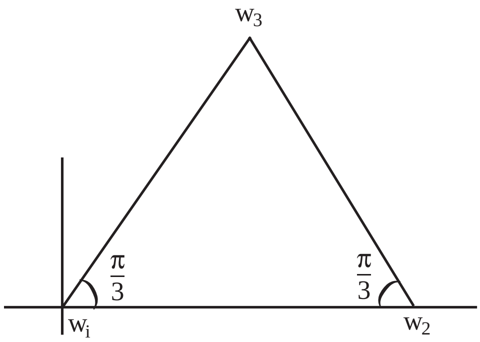


Fig. 63

It is convenient to choose three arbitrary points $x_1 = -1$, $x_2 = 1$ and $x_3 = \infty$ which are mapped into the vertices of the equilateral triangle, so we take S-C transformation (78).

$$f(z) = A\int_1^z (s+1)^{-2/3}(s-1)^{-2/3}ds$$

Here, $f(-1) = w_1 = 0$ and $f(1) = w_2 = 5$. So that

$$A = 5 / \int_{-1}^1 (s^2-1)^{-2/3}ds$$

Hence the desired transformation is

$$f(z) = \frac{5\int_1^z (s^2-1)^{2/3}ds}{\int_{-1}^1 (s^2-1)^{2/3}ds}$$

Alternative : We take $z_0 = -1$, $A = 1$, $B = 0$ and find S-C transformation as, (choosing one of x_i's as point at infinity)

$$w = \int_1^2 (s+1)(s-1)^{2/3}ds \tag{79}$$

taking $x_1 = -1$ and $x_2 = 1$.

Then $\tilde{f}(1) = \tilde{w}_2$, say, and the image of the point $z = -1$ is the point $\tilde{w}_1 = 0$. When $z = 1$ in the integral we can write $s = x$, where $-1 < x < 1$. Then $x + 1 > 0$ and Arg $(x+1) = 0$, while $|x-1| = 1-x$ and Arg $(x-1) = \pi$. Hence

$$\tilde{w}_2 = \int_{-1}^1 (x+1)^{2/3}(1-x)^{2/3}e^{-i2\pi/3}dx$$

$$= -e^{i\pi/3}\int_{-1}^{1}\frac{dx}{(1-x^2)^{2/3}} = -\,e^{i\pi/3}\int_{0}^{1}\frac{2}{(1-x^2)^{2/3}}\,dx$$

$$= -e^{i\pi/3}\int_{0}^{1}\frac{dt}{\sqrt{t}(1-t)^{2/3}},\ \text{substituting } x = \sqrt{t}.$$

$= -e^{i\pi/3}B\left(\frac{1}{2},\frac{1}{3}\right)$. We choose w_2 as, $w_2 = k\tilde{w}_2 = 5$ where $k = -5e^{-i\pi/3} \,/\, B\left(\frac{1}{2},\frac{1}{3}\right)$.

To find w_3 let us first calculate for $\tilde{w}_3$.

$$\tilde{w}_3 = \int_{-1}^{\infty}(x+1)^{-2/3}(x-1)^{-2/3}dx$$

$$= \int_{-1}^{1}(x+1)^{-2/3}(x-1)^{-2/3}dx + \int_{1}^{\infty}(x+1)^{-2/3}(x-1)^{-2/3}dx$$

$$= -e^{i\pi/3}B\left(\frac{1}{2},\frac{1}{3}\right) + e^{-i\pi}\int_{-1}^{\infty}\left(|x+1||x-1|\right)^{-2/3}dx$$

$$= -e^{-i\pi/3}B\left(\frac{1}{2},\frac{1}{3}\right) + e^{-i\pi}\int_{-i}^{\infty}\left(|x+1||x-1|\right)^{-2/3}dx$$

$$= \text{---} + e^{-i\pi+i\frac{2\pi}{3}+i\frac{2\pi}{3}}\int_{-1}^{-\infty}|x+1|^{-2/3}e^{-i\frac{2\pi}{3}}|x-1|^{-2/3}e^{-2\pi i/3}dx$$

$$= \text{---} + e^{1\pi/3}\int_{-1}^{-\infty}(x+1)^{-2/3}(x-1)^{-2/3}dx$$

Now, the value of $\tilde{w}_3$ can also be represented by the integral $\int_{-i}^{-\infty}(x+1)^{-2/3}(x-1)^{-2/3}dx$ when z tends to infinity along the negative real axis. Thus from the above relation, we have

$$\tilde{w}_3 = -e^{i\pi/3}B\left(\frac{1}{2},\frac{1}{3}\right) + e^{i\pi/3}\tilde{w}_3$$

i.e., $$\tilde{w}_3 = -e^{i\pi/3}\cdot e^{i\pi/3}B\left(\frac{1}{2},\frac{1}{3}\right)$$

So, $$w_3 = k\tilde{w}_3 = 5e^{\frac{i\pi}{3}}$$

Therefore, the three vertices of the equilateral triangle are $w_1 = 0$, $w_2 = 5$ and $w_3 = 5e^{i\pi/3}$. Clearly each of it's side is of length 5 unit. The desired transformation is then

$$f(z) = K\tilde{f}(z)$$

$$= \frac{-5e^{-i\pi/3}}{B\left(\frac{1}{2}, \frac{1}{3}\right)} \int_{-1}^{z} (s+1)^{-2/3}(s-1)^{-2/3} ds$$

which is same as obtained in the first process.

Remark : Following the above technique we can determine a S-C transformation from Im $z \geq 0$ onto a triangle, in particular, whose one side opposite to an angle is given.

Rectangle :

Example 3 : Find a S-C transformation that maps the upper half of the z-plane to the inside of the rectangle in the w-plane with vertices $-a$, a, $a + ib$ and $-a + ib$ which are the preimages of -1, 1, α and $-\alpha$ respectively.

Solution :

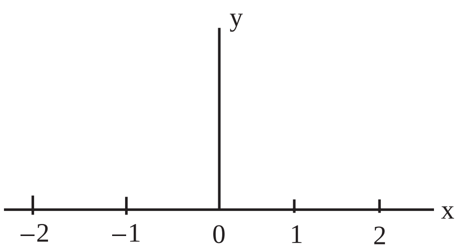


Fig. 64

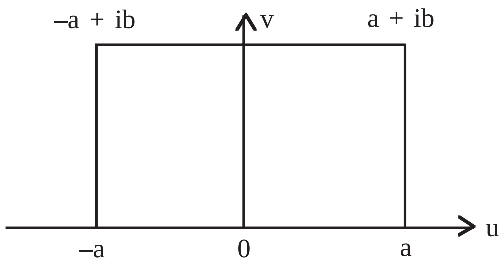


Fig. 65

Let us first make the identification of the vertices of the rectangle

$$w_1 = -a + ib,\ w_2 = -a,\ w_3 = a,\ w_4 = a+ib$$

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1/2$$

We choose

$$x_1 = -\alpha,\ x_2 = -1,\ x_3 = 1,\ x_4 = \alpha$$

where $\alpha > 1$ will be determined later. We are attempting to benefit from the symmetry here, which requires the image $z = 0$ to be $w = 0$. So taking $z_0 = 0$ we get $B = 0$ in the formula (74) for S-C transformation, which reduces to

$$f(z) = A\int_{0}^{z} [s+\alpha)(s+1)(s-1)(s-\alpha)]^{-1/2} ds$$

$$= A\int_0^z \frac{ds}{\sqrt{[(1-s^2)(\alpha^2-s^2)]}} (\equiv \phi(z,\alpha)) \tag{80}$$

The constant A may be found by using the fact that f(1) = a i.e.,

$$a = A\int_0^1 \frac{ds}{\sqrt{[(1-s^2)(\alpha^2-s^2)]}} \text{ or } A = a \Big/ \int_0^1 \frac{ds}{\sqrt{[(1-s^2)(\alpha^2-s^2)]}}$$

$$= a/\phi(\alpha), \text{ say} \tag{81}$$

To find α, we apply $f(\alpha) = a + ib$,

$$a + ib = \frac{a}{\phi(\alpha)} \int_0^a \frac{ds}{\sqrt{[(1-s^2)(\alpha^2-s^2)]}}$$

$$= \frac{a}{\phi(\alpha)} \left\{ \int_0^1 \frac{ds}{\sqrt{[(1-s^2)(\alpha^2-s^2)]}} + i\int_1^\alpha \frac{ds}{\sqrt{[(s^2-1)(\alpha^2-s^2)]}} \right\}$$

from which, equating imaginary parts, we arrive at

$$b\phi(\alpha) = \alpha\int_1^\alpha \frac{ds}{\sqrt{[(s^2-1)(\alpha^2-s^2)]}}$$

Since a and b are known, this equation determines α, which gives rise to the evaluation of $\phi(\alpha)$ i.e. A is completely known.

Note : The function $\phi(z, \alpha)$, given in (80), which involves z as the upper limit of an integral, is called an **elliptic integral of the first kind** and it is not an elementary function. The real definite integral $\phi(\alpha)$ in (81) is called a complete elliptic integral of the first kind.

Example 4 : Find a Schwarz-Christoffel transformation that maps the upper half of the z-plane to the vertical semi-infinite strip $-\pi/2 < u < \pi/2$, $v > 0$ of the w-plane.

Solution :

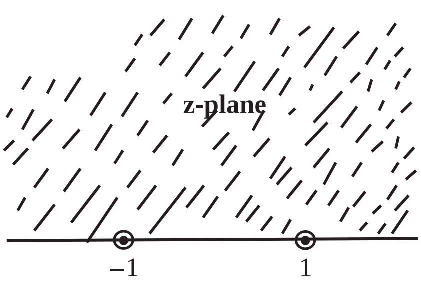


Fig. 66

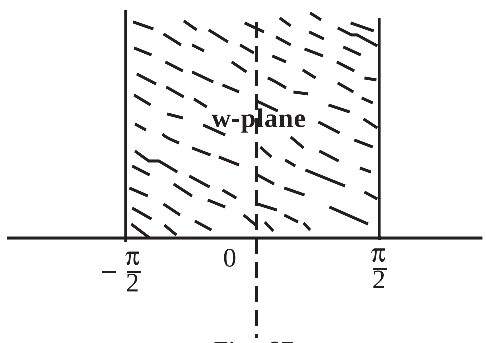


Fig. 67

Here we take $x_1 = -1$, $x_2 = 1$ and $x_3 = \infty$ and the image points are $w_1 = -\pi/2$ and $w_2 = \pi/2$ respectively, so that a S-C transformation can be written as

$$f(z) = A\int_{z_0}^{z} (s+1)^{-1/2}(s-1)^{-1/2}\,ds + B$$

$$= A\int_{z_0}^{z} \frac{1}{(s^2-1)^{1/2}}\,ds + B$$

$$= \tilde{A}\log\left(iz\sqrt{1-z^2}\right) + \tilde{B}$$

Using $f(-1) = -\dfrac{\pi}{2}$ and $f(1) = \dfrac{\pi}{2}$, we find

$$f(z) = -i\log\left(iz + \sqrt{1-z^2}\right),$$

Choosing a suitable branch of the logarithm.

Unit 6 ❑ Entire and Meromorphic Functions

Structure

6.0 The Objectives of the Chapter

In this chapter we shall study entire functions, their growth properties and meromorphic functions. Infinite products and their convergence will be discussed. Properties of zeros of an entire function, convex functions, gamma function and its important properties will also be discussed.

6.1 Entire function

A function f(z) analytic in the finite complex plane is said to be entire (or sometimes integral) function. Clearly, the sum, difference and product of two or more entire functions are entire functions.

Examples : The polynomial function $P(z) = a_0 + a_1 z + ... + a_n z^n$, exponential function e^z, sin z, cos z etc. are entire functions.

Let us consider the first example, the polynomial function. It is evident that P(z) can be uniquely expressed as a product of linear factors in the form

$$A_0\left(1-\frac{z}{z_1}\right)\left(1-\frac{z}{z_2}\right)\cdots\left(1-\frac{z}{z_n}\right), \text{ if } a_0 \neq 0$$

or,

$$A_p z^p\left(1-\frac{z}{\varsigma_1}\right)\left(1-\frac{z}{\varsigma_2}\right)\cdots\left(1-\frac{z}{\varsigma_{n-p}}\right), \text{ if } a_0 = a_1 = \cdots a_{p-1} = 0,\ a_p \neq 0, \qquad (82)$$

where A_0(or, A_p) is constant and $z = z_1, z_2, ..., z_n$(or, $z = 0, \varsigma_1, \varsigma_2, ..., \varsigma_{n-p}$) are the zeros of P(z), multiple zeros are counted according to their multiplicities. There arises a natural question : whether any entire function can be expressed in a similar manner in terms of its zeros. The observations are as follows :

(i) There may exist entire function which never vanishes,

(ii) If an entire function possesses finite number of zeros, then it is always possible to express it in the form (82) stated above. But when the number of zeros are infinite the form (82) reduces to a product of infinite number of linear factors which need not always be convergent. We first consider infinite products of complex numbers and functions.

6.2 Infinite Products

An infinite product is an expression of the form

$$\prod_{n=1}^{\infty} p_n \qquad (83)$$

where $p_1, p_2, ..., p_n, ...$ are non-zero complex factors. If we allow any of the factors be zero, it is evident that the infinite product would be zero regardless of the behaviour of the other terms.

Let $\quad P_n = p_1 p_2 ... p_n.$

If P_n tends to a finite limit (non-zero) p as n tends to infinity, we say that the infinite product (83) is convergent and write as

$$\prod_{n=1}^{\infty} p_n = p \tag{84}$$

An infinite product which does not tend to a non-zero finite limit as n tends to infinity is said to be divergent.

To find the necessary condition for convergence for the infinite product $\prod_{n=1}^{\infty} p_n$, say (84) holds, then writing p_n as

$$p_n = \frac{P_n}{P_{n-1}}$$

we conclude in view of (84) that $\lim_{n\to\infty} p_n = \lim_{n\to\infty} \frac{P_n}{P_{n-1}} = \frac{P}{P} = 1$

Thus, $$\lim_{n\to\infty} p_n = 1 \tag{85}$$

is a necessary condition for convergence of the infinite product (83). It is then better to write the product as

$$\prod_{n=1}^{\infty} (1 + a_n) \tag{86}$$

so that $a_n \to 0$ as $n \to \infty$ is a necessary condition for convergence.

Theorem 6.1 : The infinite product (86) converges if and only if

$$\sum_{n=1}^{\infty} \log(1 + a_n) \tag{87}$$

converges. We use the principal branch of the log function and omit, as usual, the terms with $a_n = -1$.

Proof. Let $P_n = \prod_{k=1}^{n} (1 + a_k)$ and $S_n = \sum_{k=1}^{n} \log(1 + a_k)$.

Then $\log P_n = S_n$ and $P_n = e^{S_n}$. Now if the given series is convergent i.e. $S_n \to S$ as $n \to \infty$, P_n tends to the limit $P = e^S$ $(\neq 0)$. This proves the sufficiency of the condition.

Conversely, assume that the product converges i.e. $P_n \to P\ (\neq 0)$ as $n \to \infty$. We shall show, by virtue of $P_n = e^{S_n}$, that the series (87) converges to some value of log P, not necessarily the principal value of log P.

For $n \to \infty$, $\frac{P_n}{P} \to 1$ and $\text{Log}\left(\frac{P_n}{P}\right) \to 0.$

Now there exists an integer K_n such that

$$\text{Log}\left(\frac{P_n}{P}\right) = S_n - \text{Log}\,P + 2k_n\pi i \tag{88}$$

To establish the convergence of the sequence $\{k_n\}$, we form the difference

$$(k_{n+1} - k_n)2\pi i = \text{Log}\left(\frac{P_{n+1}}{P}\right) - \text{Log}\left(\frac{P_n}{P}\right) - \text{Log}(1 + a_{n+1})$$

$$= i\left\{\text{Arg}\left(\frac{P_{n+1}}{P}\right) - \text{Arg}\left(\frac{P_n}{P}\right) - \text{Arg}(1 + a_{n+1})\right\}$$

and that

$$k_{n+1} - k_n = \frac{1}{2\pi}\left\{\text{Arg}\left(\frac{P_{n+1}}{P}\right) - \text{Arg}\left(\frac{P_n}{P}\right) - \text{Arg}(1 + a_{n+1})\right\}$$

tends to zero as $n \to \infty$, and let the limit of the sequence $\{k_n\}$ be k.

Taking limit in (88), we find that

$$S_n \to \text{Log}P - 2k\pi i$$

and so the condition assumed is necessary.

Definition : An infinite product $\prod_{n=1}^{\infty}(1 + a_n)$ is absolutely convergent if and only if $\sum_{n=1}^{\infty}|\log(1 + a_n)|$ is convergent.

Theorem 6.2 : The infinite product (86) converges absolutely if and only if the series Σa_n converges absolutely.

Proof : If Σa_n converges absolutely, then in particular $a_n \to 0$ as $n \to \infty$. Also, if $\sum_{n=1}^{\infty}\log(1 + a_n)$ converges absolutely then $\log(1 + a_n) \to 0$ and $a_n \to 0$. Thus in either of the cases $a_n \to 0$ and we can take $|a_n| \leq \frac{1}{2}$ for sufficiently large n. Then by elementary calculation,

$$\left|1 - \frac{\log(1+a_n)}{a_n}\right| = \left|\frac{a_n}{2} - \frac{a_n^2}{3} + \cdots\right|$$

$$\leq \frac{1}{2}\left\{|a_n| + |a_n|^2 + |a_n|^3 + \cdots\right\} \leq \frac{1}{2},$$ n = large enough. It follows that

$$\frac{1}{2}|a_n| \leq \log|(1+a_n)| \leq \frac{3}{2}|a_n|$$

confirming the occurrence of the absolute convergence simultaneously for the two series.

6.3 Infinite product of functions

So far we have considered infinite product of complex numbers. Now we shall study infinite products whose factors are functions of a complex variable. Some of the factors (finite in number) may vanish on a region considered. In that case we consider the infinite product omitting those factors. The theorems proved earlier hold good in this case too with some modifications.

Definition : (Uniform convergence of infinite products)

An infinite product

$$\prod_{n=1}^{\infty}\{1 + a_n(z)\} \tag{89}$$

where the functions $a_n(z)$ are defined on a region D, is said to be uniformly convergent on D if the sequence of partial products

$$P_n(z) = \prod_{k=1}^{n}\{1 + a_k(z)\}$$

converges uniformly to a non-zero limit on D.

Theorem 6.3 : An infinite product (89) is uniformly convergent on a domain D if the series $\sum_{n=1}^{\infty}|a_n(z)|$ converges uniformly and has a bounded sum there.

Proof : Let M be the upper bound of the sum $\sum|a_n(z)|$ on D. Then

$$\{|1 + a_1(z)|\}\ \{1 + |a_2(z)|\} \ldots \{1 + |a_n(z)|\} < e^{|a_1(z)| + |a_2(z)| + \ldots |a_n(z)|} \leq e^M$$

Let us consider the sequence $\{Q_n\}$ with

$$Q_n(z) = \prod_{k=1}^{n} \{1+|a_k(z)|\}$$

We observe

$$Q_n(z) - Q_{n-1}(z) = \{1+|a_1(z)|\}\{1+|a_2(z)|\}\ldots\{1+|a_{n-1}(z)|\}|a_n(z)|$$

$$< e^M |a_n(z)|$$

Now since the series $\sum|a_n(z)|$ is uniformly convergent, the series $\sum\{Q_n(z) - Q_{n-1}(z)\}$ is uniformly convergent. Thus the sequence $\{Q_n\}$ tends to a limit. Again

$$|P_n(z) - P_{n-1}(z)| \le Q_n(z) - Q_{n-1}(z),$$

so the result follows.

Theorem 6.4 : An infinite product $\prod_{n=1}^{\infty}\{1+a_n(z)\}$ converges uniformly and absolutely in a closed bounded domain D if each function $a_n(z)$ satisfies $|a_n(z)| \le M_n$ for all z ε D and M_n is independent of z and moreover ΣM_n is convergent.

Proof : Given ΣM_n is convergent, so the infinite product $M = \prod_{n=1}^{\infty}(1+M_n)$ converges by theorem 6.2

Now, for n > m

$$|Q_n(z) - Q_m(z)| = |Q_m(z)|\left|\prod_{m+1}^{n}\{1+a_k(z)\} - 1\right| \qquad (90)$$

Again,

$$\prod_{m+1}^{n}\{1+a_k(z)\} - 1 = \sum_{k=m+1}^{n} a_k(z) + \sum_{i,j}^{n} a_i(z)a_j(z) + \sum_{i,j,l}^{n} a_i(z)a_j(z)a_l(z)$$

$$+\ldots+ a_{m+1}(z)a_{m+2}(z)\ldots a_n(z).$$

Taking moduli

$$\left|\prod_{m+1}^{n}\{1+a_k(z)\} - 1\right| \le \sum_{k=m+1}^{n} M_k + \sum_{i,j}^{n} M_i M_j + \sum_{i,j,l}^{n} M_i M_j M_l +$$

$$+\ldots+ M_{m+1}M_{m+2}\ldots M_n$$

$$= \prod_{m+1}^{n}(1+M_k) - 1$$

Utilising this in (90) we obtain

$$|Q_n(z) - Q_m(z)| \le \prod_{k=1}^{m}(1+M_k)\left\{\prod_{m=1}^{n}(1+M_k) - 1\right\}$$

$$= \prod_{k=1}^{n}(1+M_k) - \prod_{k=1}^{m}(1+M_k) \qquad (91)$$

Now as the infinite product $\prod_{1}^{\infty}(1+M_k)$ is convergent, we choose m large enough so that r.h.s in (91) is less than ε and hence

$$|Q_n(z) - Q_m(z)| < \varepsilon, \text{ when } n > m$$

Thus the sequence $\{Q_n(z)\}$ converge uniformly, since m depends only on ε.

Finally, absolute convergence of the infinite product follows on utilising Th. 6.2

Example 1 : Test for convergence of the infinite product

$$\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

Solution : The terms of the product vanish when $z = \pm 1, \pm 2, \ldots$ etc.

Here $a_n(z) = -\frac{z^2}{n^2}$ and $|a_n(z)| \le |z^2| \frac{1}{n^2}$

Now since the series $\sum \frac{1}{n^2}$ is convergent, the given infinite product is uniformly and absolutely convergent in the entire plane excluding the points $z = \pm 1, \pm 2,$ etc.

Example 2 : Discuss the convergence of the infinite product

$$\left(1 - \frac{z}{1}\right)\left(1 + \frac{z}{1}\right)\left(1 - \frac{z}{2}\right)\left(1 + \frac{z}{2}\right)\cdots$$

Solution : Let $P_n(z) = \prod_{k=1}^{n}\left(1 - \frac{z^2}{k^2}\right)$ and we consider a bounded closed domain D which does not contain the points $z = \pm 1, \pm 2, \ldots$. The sequence $\{P_n(z)\}$ converges uniformly in D (see example 1). Again let

$$F_{2n}(z) = \left(1 - \frac{z}{1}\right)\left(1 + \frac{z}{1}\right)\left(1 - \frac{z}{2}\right)\left(1 + \frac{z}{2}\right)\cdots\left(1 - \frac{z}{n}\right)\left(1 + \frac{z}{n}\right)$$

$$F_{2n+1}(z) = F_{2n}(z)\left(1 - \frac{z}{n+1}\right),$$

then $\quad F_{2n}(z) = P_n(z)$ and $F_{2n+1}(z) = \left(1 - \frac{z}{n+1}\right)P_n(z)$

and obviously the sequences F_2, F_4, F_6, ... and F_1, F_3, F_5 ... converge uniformly in D. Hence the given infinite product converges uniformly in D.

To test for the absolute convergence of the given product we notice that

$$\sum_{i}^{\infty} |a_n| = |z|\left\{1+1+\frac{1}{2}+\frac{1}{2}+\frac{1}{3}+\frac{1}{3}+\cdots\right\}$$

and it is divergent since the series on the right is divergent and |z| is finite. Therefore the given product does not converge absolutely.

Considering the theorem 4.4 on uniformly convergent sequence of analytic functions [(14) Page-72] we get the following theorem :

Theorem 6.5 : If an infinite product $\Pi\{1 + f_n(z)\}$ converges uniformly to f(z) in a bounded closed domain D and if each function $f_n(z)$ is analytic in D, then f(z) is also analytic in D.

6.4 Weierstrass' Factorization

Theorem 6.6 : If f(z) is an entire function and never vanishes on $\mathbb{C}$, then f(z) is of the form $f(z) = e^{g(z)}$, or, more generally, $f(z) = ce^{g(z)}$, $c \neq 0$, constant.

where g(z) is also an entire function.

Proof : Since f is entire and never vanishes on $\mathbb{C}$, f^1/f is also entire and is thus the derivative of an entire function g(z). [follows from Result 1, PG(MT) 02-complex analysis [14, page-54]. Then

$$\frac{f'}{f} = g'$$

i.e. $\quad f' = fg'$

Now, $\quad (fe^{-g})' = f'e^{-g} - fg'e^{-g} = 0$

Hence, $\quad f(z) = ce^{g(z)}$ proving the result.

Assume now that f possesses finitely many zeros, a zero of order m > 0 at the origin, and the non-zero ones, possibly repeated are a_1, ... a_n. Then

$$f(z) = z^m \prod_{k=1}^{n}\left(1-\frac{z}{a_n}\right)e^{g(z)}$$

where g is entire.

This is clear, since if we divide f by the factors which produce zero at the points z = 0, a_1, ..., a_n we get an entire function with no zeros.

However we cannot expect, in general, such a simple formula to hold in the case of infinitely many zeros. Here we have to take care of convergence problems for an infinite product. In fact the obvious generalization.

$$f(z) = z^m \prod_{k=1}^{n} \left(1 - \frac{z}{a_k}\right) e^{g(z)}$$

is valid in a bounded closed domain D if the infinite product converges uniformly in D.

Theorem 6.7 (Weierstrass' Factorization Theorem) :—

Let $\{a_n\}$ be a sequence of complex numbers with the property $a_n \to \infty$ as $n \to \infty$. Then it is possible to construct an entire function f(z) with zeros precisely at these points.

Proof : We need Weierstrass' primary factors to construct the desired function. The expressions $E(z, o) = 1 - z$, $E(z, p) = (1-z)e^{z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}}$, $p = 1, 2$ ..., are called Weierstrass' primary factors. Each primary factor is an entire function having only one simple zero at $z = 1$.

Now, when $|z| < 1$ we have, $\log E(z, p) = \log (1-z) + z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}$

$$= \left(-z - \frac{z^2}{2} - \ldots - \frac{z^p}{p} - \frac{z^{p+1}}{p+1} - \ldots\right) + \left(z + \frac{z^2}{2} + \ldots + \frac{z^p}{p}\right) = -\frac{z^{p+1}}{p+1} - \frac{z^{p+2}}{p+2} - \ldots$$

Here we have taken the principal branch of $\log (1 - z)$.
Hence if

$$|z| \le \frac{1}{2}, \; |\log E(z,p)| \le |z|^{p+1} + |z|^{p+2} + \ldots = |z|^{p+1}\left(1 + |z| + |z^2| + \ldots\right)$$

$$\le |z|^{p+1}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\right) = 2|z|^{p+1} \ldots \tag{92}$$

We may suppose that the origin is not a zero of the entire function f(z) to be constructed so that $a_n \neq 0$ for all n.

For, if origin is a zero of f(z) of order m we need only multiply the constructed function by z^m. We also arrange the zeros in order of non-decreasing modulus (if several distinct points a_n have the same modulus, we take them in any order) so that $|a_1| \le |a_2| \le \ldots$. Let $|a_n| = r_n$.

Since $r_n \to \infty$ we can always find a sequence of positive inegers

$m_1, m_2, \ldots m_n, \ldots$ such that the series $\sum_{n=1}^{\infty} \left(\frac{r}{r_n}\right)^{m_n}$ converges for all positive values of r.

In fact, we may take $m_n = n$ since for any given value of r, we have $\left(\frac{r}{r_n}\right)^n < \frac{1}{2^n}$ for all sufficiently large n and the series is therefore convergent. Next we take an arbitrary positive number R and choose the integer N such that $r_N \le 2R < r_{N+1}$. Hence, when $|z| \le R$ and $n > N$ we have,

$$\left|\frac{z}{a_n}\right| \le \frac{R}{r_n} \le \frac{R}{r_{N+1}} < \frac{1}{2} \text{ and so by (92),}$$

$\left|\log E\left(\frac{z}{a_n}, m_n\right)\right| \le 2\left|\frac{R}{r_n}\right|^{m_n+1}$ By Weierstrass' M-test the series $\sum_{n=1}^{\infty} \log E\left(\frac{z}{a_n}, m_n\right)$ converges absolutely and uniformly when $|z| \le R$ and so the infinite product $\prod_{n=1}^{\infty} E\left(\frac{z}{a_n}, m_n\right)$ converges absolutely and uniformly in the disc $|z| \le R$, however large R may be. Hence the above product represents an entire function, say G(z).

Thus, $$G(z) = \prod_{n=1}^{\infty} E\left(\frac{z}{a_n}, m_n\right) \tag{93}$$

With the same value of R, we choose another integer k such that $r_k \le R < r_{K+1}$.

Then each of the functions of the sequence $\prod_{n=1}^{m} E\left(\frac{z}{a_n}, m_n\right), m = k+1, k+2, \ldots,$ vanish at the points a_1 ..., a_k and nowhere else in $|z| \le R$. Hence by Hurwitz's theoroem the only zeros of G in $|z| \le R$ are a_1, ... a_k. Since R is arbitrary, this implies that the only zeros of G are the points of the sequence $\{a_n\}$.

Now, if origin is a zero of order m of the required entire function f(z), then f(z) is of the form $f(z) = z^m G(z)$. Again, for any entire function g(z), $e^{g(z)}$ is also an entire function without any zero. Hence the general form of the required entire function f(z) is

$$f(z) = z^m e^{g(z)} G(z)$$

$$= z^m e^{g(z)} \prod_{n=1}^{\infty} E\left(\frac{z}{a_n}, m_n\right) \tag{94}$$

$$= z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \ldots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}} \tag{95}$$

Remark : As there are many possible sequences $\{m_n\}$ in the construction of the function G(z) and ultimately of f(z), the form of the function f(z) achieved is not unique.

6.5 Counting zeros of analytic functions

The rate of growth of an entire function is closely related to the density of zeros. We have a quite effective formula in this regard due to J.L.W.V. Jensen, a Danish mathematician who discovered it in the year 1899.

Theorem 6.8 [Jensen's Formula] :—

Let f(z) be analytic on $|z| \leq R$, $f(0) \neq 0$ and $f(z) \neq 0$ on $|z| = R$. If $a_1, ..., a_n$ be the zeros of f(z) within the circle $|z| = R$, multiple zeros being repeated according to their multiplicities, then

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta - \sum_{k=1}^{n} \log\left(\frac{R}{|a_k|}\right) \cdots \quad (96)$$

Proof : Let $\phi(z) = f(z).\prod_{k=1}^{n} \frac{R^2 - \bar{a}_k z}{R(z - a_k)} \cdots$ (97)

The zeros of the denominator of $\phi(z)$ are also the zeros of f(z) of the same order. Hence the zeros of f(z) cancels the poles a_n in the product and so $\phi(z)$ is analytic on $|z| \leq R$. Also, $\phi(z) \neq 0$ on $|z| \leq R$. For, if $R^2 - \bar{a}_k z = 0$ then $z = \frac{R^2}{\bar{a}_k}$ is the inverse point of a_k with respect to the circle $|z| = R$ and so lies outside the circle. Again,

$|\phi(z)| = |f(z)| \left|\frac{R^2 - \bar{a}_1 z}{R(z - a_1)}\right| \cdots \left|\frac{R^2 - \bar{a}_n z}{R(z - a_n)}\right|$. Now, when $|z| = R$

we have, $\left|\frac{R^2 - \bar{a}_k z}{R(z - a_k)}\right| = \left|\frac{z\bar{z} - \bar{a}_k z}{R(z - a_k)}\right| = \frac{|z|}{R}\left|\frac{\bar{z} - \bar{a}_k}{z - a_k}\right| = 1$

Hence, $|\phi(z)| = |f(z)|$ on $|z| = R$.

Since $\phi(z)$ is analytic and non-zero on $|z| \leq R$, $\log \phi(z)$ is also analytic on $|z| \leq R$ and consequently $\text{Re} \log \phi(z) = \log |\phi(z)|$ is harmonic on $|z| \leq R$. Hence by Gauss' mean value theorem,

$$\log|\phi(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|\phi\left(Re^{i\theta}\right)\right| d\theta \quad (98)$$

From (97) we have, $|\phi(0)| = |f(0)| \frac{R}{|a_1|} \cdot \frac{R}{|a_2|} \cdots\cdots \frac{R}{|a_n|}.$

Hence from (98) we get,

$$\log|f(0)| + \sum_{k=1}^{n} \log\left(\frac{R}{|a_k|}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log|\phi(Re^{i\theta})|\, d\theta$$

i.e. $\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta - \sum_{k=1}^{n} \log\left(\frac{R}{|a_k|}\right)$

(since $|\phi(z)| = |f(z)|$ on $|z| = R$)

Note : We observe that Jensen's formula can also be expressed as

$$\log \frac{R^n}{|a_1 \dots a_n|} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| d\theta - \log|f(0)| \ldots\ldots \quad (99)$$

or as, $$\log \frac{R^n}{r_1 \dots r_n} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| d\theta - \log|f(0)| \ldots\ldots \quad (100)$$

where $|a_i| = r_i$, $i = 1, \ldots, n$.

Theorem 6.9 (Jensen's inequality) :— Let f(z) be analytic on $|z| \le R$, $f(0) \ne 0$ and $f(z) \ne 0$ on $|z| = R$. If $a_1, \ldots, a_n$ be the zeros of f(z) within $|z| = R$, multiple zeros being repeated according to their multiplicities, and $|a_i| = r_i$, $i = 1, \ldots, n$, then

$$\frac{R^n |f(0)|}{r_1 \dots r_n} \le M(R) \quad (101)$$

where $M(R) = \max_{|z|=R} |f(z)|.$

Proof : As in Jensen's formula (theorem 6.8) we have, $|\phi(z)| = |f(z)|$ on $|z| = R$ and so by the maximum modulus theorem, $|\phi(z)| \le M(R)$ for $|z| \le R$. In particular,

$|\phi(0)| \le M(R)$

i.e. $\frac{R^n |f(0)|}{r_1 \dots r_n} \le M(R).$

Theorem 6.10 (Poisson-Jensen formula) :- Let f(z) be analytic on $|z| \le R$, $f(0) \ne 0$ and $f(z) \ne 0$ on $|z| = R$. If $a_1 \ldots a_n$ be the zeros of f(z) within the circle $|z| = R$, multiple zeros being repeated according to the their multiplicities, then for any $z = re^{i\theta}$, $r < R$,

$$\log|f(re^{i\theta})| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(t-\theta)} \log|f(Re^{it})| dt - \sum_{k=1}^{n} \log\left|\frac{R^2 - \bar{a}_k re^{i\theta}}{R(re^{i\theta} - a_k)}\right|.$$

Proof : Let $\phi(z) = f(z).\prod_{k=1}^{n} \frac{R^2 - \bar{a}_k z}{R(z - a_k)}$. Then, as in Jensen's formula we have, $|\phi(z)| = |f(z)|$ on $|z| = R$. Since $\phi(z)$ is analytic and non-zero on $|z| \le R$, log $\phi(z)$ is also analytic on $|z| \le R$ and consequently log $|\phi(z)|$ is harmonic on $|z| \le R$.

So, by Poisson's integral formula,

$$\log\left|\phi(re^{i\theta})\right| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(t-\theta)} \log\left|\phi(Re^{it})\right| dt \qquad (102)$$

Now, $\log\left|\phi(re^{i\theta})\right| = \log\left|f(re^{i\theta})\right| + \sum_{k=1}^{n} \log\left|\frac{R^2 - \bar{a}_k re^{i\theta}}{R(re^{i\theta} - a_k)}\right|$

Since $\log|\phi(z)| = \log|f(z)|$ on $|z| = R$ we get from (102)

$$\log\left|f(re^{i\theta})\right| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(t-\theta)}.\log\left|f(Re^{it})\right| dt$$

$$-\sum_{k=1}^{n} \log\left|\frac{R^2 - \bar{a}_k re^{i\theta}}{R(re^{i\theta} - a_k)}\right| \qquad (103)$$

6.6 Convex functions

The property of convexity plays an important role in function theory because in several cases some lead factors associated with entire, meromorphic and subharmonic functions appear to be convex functions.

A real-valued function ϕ defined on the interval I = [a, b] is said to be convex if for any two points s, u in [a, b]

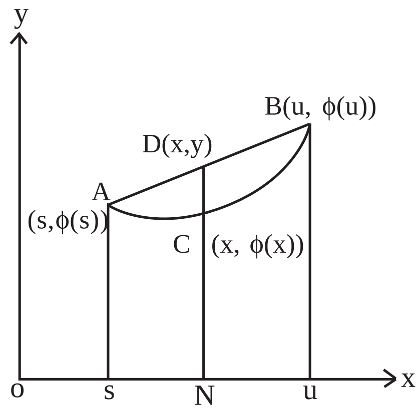
$$\phi(\lambda u + (1-\lambda)s \le \lambda\phi(u) + (1-\lambda)\,\phi\,(s) \text{ for } 0 \le \lambda \le 1 \qquad (104)$$

Geometrically, the condition (104) is equivalent to the condition that if s < x < u, then the point (x, ϕ(x)) should lie below or on the chord joining the points (s, ϕ(s)) and (u, ϕ(u)) in the plane.

Analytical condition for ϕ(x) to be convex in [a, b] :- Let the coordinates of the points A, B, C on the curve y = ϕ(x) as shown in the adjoining figure be (s, ϕ(s)), (u, ϕ(u)) and (x, ϕ(x)) respectively where s < x < u.

Equation of the chord AB is $y - \phi(x) = \dfrac{\phi(u) - \phi(s)}{u - s}(x - s)$.

or, $y = \phi(s) + \dfrac{\phi(u) - \phi(s)}{u - s}(x - s)$ (105)



Let the coordinates of any point D on the chord AB be (x, y). According to definition $\phi(x)$ will be convex if and only if $CN \leq DN$. i.e., if and only if $\phi(x) \leq y$; i.e. if and only if

$\phi(x) \leq \phi(s) + \dfrac{\phi(u) - \phi(s)}{u - s}(x - s)$; i.e., if and only if

$$\phi(x) \leq \frac{u - x}{u - s}\phi(s) + \frac{x - s}{u - s}\phi(u) \qquad (106)$$

for $s < x < u$.

We now state two results on convex functions without proof.

Result 1. A differentiable function f(x) on [a, b] is convex if and only if $f'(x)$ is increasing in [a, b].

Result 2. A sufficient condition for f(x) to be convex is that $f''(x) > 0$.

The maximum modulus function : Let f(z) be a non-constant analytic function in $|z| < R$. Then for $0 \leq r < R$ we define the maximum modulus function M(r, f) or, simply M(r) by $M(r) = \max\limits_{|z|=r} |f(z)|$. By maximum modulus theorem we can also write $M(r) = \max\limits_{|z|=r} |f(z)|$.

Result : Let f(z) be a non-constant analytic function in $|z| < R$. Then M(r) is a strictly increasing function of r in $0 \leq r \leq R$.

Proof : Let $0 \leq r_1 < r_2 < R$. Since f(z) is analytic in $|z| \leq r_2$, the maximum value of $|f(z)|$ for $|z| \leq r_2$ is attained on $|z| = r_2$. Let z_2 be a point on $|z| = r_2$ such that $|f(z_2)| = M(r_2)$. Similarly, the maximum value of $|f(z)|$ for $|z| \leq r_1$ is attained on $|z| = r_1$. Let z_1 be a point on $|z| = r_1$ such that $|f(z_1)| = M(r_1)$.

Since $r_1 < r_2$, z_1 is an interior point of the closed region $|z| \leq r_2$. Hence by maximum modulus theorem,

$|f(z_1)| < M(r_2)$; i.e. $M(r_1) < M(r_2)$.

This proves the result.

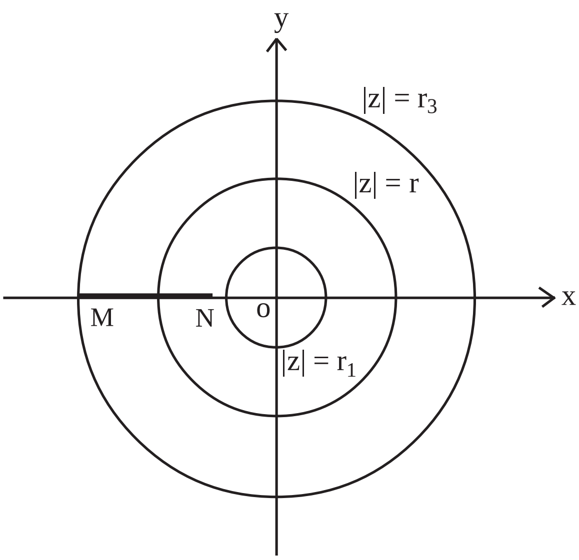
Corollary : Let f(z) be a non-constant entire function. Then its maximum modulus function $M(r) \to \infty$ as $|z| = r \to \infty$. For, if M(r) is bounded, then by Liouville's theorem f(z) would be a constant function.

Theorem 6.11 [Hadamard's three-circles theorem].

Let $0 < r_1 < r < r_3$ and suppose that f(z) is analytic on the closed annulus $r_1 \le |z| \le r_3$. If $M(r) = \max_{|z|=r} |f(z)|$, , then

$$M(r)^{\log\left(\frac{r_3}{r_1}\right)} \le M(r_1)^{\log\left(\frac{r_3}{r}\right)} . M(r_3)^{\log\left(\frac{r}{r_1}\right)} \qquad (107)$$

Proof : Let us consider the function $\phi(z) = z^{\alpha} f(z)$, where α is a real constant to be chosen later. If $\alpha \ne$ an integer, $\phi(z)$ is multi-valued in $r_1 \le |z| \le r_3$ and so we cut the annulus along the negative part of the real axis. Thus we obtain a simply connected region G in which the principal branch of $\phi(z)$ is analytic. Hence the maximum modulus of this branch of $\phi(z)$ in G is attained on the boundary of G. Since α is real, all the branches of $\phi(z)$ have the same modulus. If we consider another branch of $\phi(z)$ which is analytic in another cut annulus it is clear that the principal branch of $\phi(z)$ can not attain its maximum value on the cut. Hence maximum of $|\phi(z)|$ is attained on at least one of the bounding circles $|z| = r_1$ or, $|z| = r_3$. Thus,



$\left|z^{\alpha} f(z)\right| \le \max\left(r_1^{\alpha} M(r_1), r_3^{\alpha} M(r_3)\right)$. Hence on $|z| = r$,

$$r^{\alpha} M(r) \le \max\left(r_1^{\alpha} M(r_1), r_3^{\alpha} M(r_3)\right) \qquad (108)$$

We now choose α such that $r_1^{\alpha} M(r_1) = r_3^{\alpha} M(r_3)$. Then

$\alpha = -\dfrac{\log(M(r_3))/M(r_1))}{\log(r_3/r_1)}$. Substituting this value of α in (108) we get,

$$M(r) \le \left(\frac{r}{r_1}\right)^{-\alpha} M(r_1)$$

$$= \left(\frac{r}{r_1}\right)^{\log\left(\frac{M(r_3)}{M(r_1)}\right)} \bigg/ \log\left(\frac{r_3}{r_1}\right) . M(r_1)$$

and so $\quad M(r)^{\log(r_3/r_1)} \le \left(\frac{r}{r_1}\right)^{\log(M(r_3)/M(r_1))} . M(r_1)^{\log(r_3/r_1)}$

That is, $\quad M(r)^{\log(r_3/r_1)} \le \left(\frac{M(r_3)}{M(r_1)}\right)^{\log(r/r_1)} . M(r_1)^{\log(r_3/r_1)}$ [since $a^{\log b} = b^{\log a}$]

$= M(r_1)^{\log(r_3/r)} . M(r_3)^{\log(r/r_1)}$.

Note : Equality in (107) occurs when $\phi(z)$ is a constant, i.e. when f(z) is of the form cz^{α} for some real α and c is a constant.

Corollary : log M(r) is a convex function of log r.

Proof : Let f(z) be analytic in the closed annulus $0 < r_1 \le |z| \le r_2$.

If $r_1 < r < r_2$ we have, by Hadamard's three-circles theorem,

$M(r)^{\log(r_2/r_1)} \le M(r_1)^{\log(r_2/r)} . M(r_2)^{\log(r/r_1)}$. Taking logarithms we get

$(\log r_2 - \log r_1) \log M(r) \le (\log r_2 - \log r) \log M(r_1) +$

$(\log r - \log r_1) \log M(r_2)$. That is,

$$\log M(r) \le \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log M(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log M(r_2) \qquad (109)$$

The inequality (109) shows that log M(r) is a convex function of log r.

6.7 Order of an entire function

An entire function f(z) is said to be of finite order if there is a positive number A such that as $|z| = r \to \infty$, the inequality $M(r) < e^{r^A}$ holds.

The lower bound ρ of such numbers A is called the order of the function.

f is said to be of infinite order if it is not of finite order. From the definition it is clear that order of an entire function is non-negative.

Result : Let f be an entire function of order ρ and $M(r) = \max\{|f(z)| : |z| = r\}$. Then

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} \tag{110}$$

Proof : By hypothesis, given $\varepsilon > 0$ there exists $r_0(\varepsilon) > 0$ such that

$$M(r) < e^{r^{\rho+\varepsilon}} \text{ for } r > r_0$$

while $M(r) > e^{r^{\rho+\varepsilon}}$ for an increasing sequence $\{r_n\}$ of values of r, tending to infinity. In otherwords,

$$\frac{\log\log M(r)}{\log r} < \rho + \varepsilon \quad \forall r > r_0 \text{ and} \tag{111}$$

$$\frac{\log\log M(r)}{\log r} > \rho - \varepsilon \tag{112}$$

for a sequence of values of $r \to +\infty$

(111) and (112) precisely means

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}$$

Example 3 : Determine the order of the functions.

(i) $p(z) = a_0 + a_1 z + ... + a_n z^n$, $a_n \neq 0$. (ii) e^{kz}, $k \neq 0$.

(iii) sin z (iv) $\cos\sqrt{z}$

Solution :

(i) $|p(z)| = |a_0 + a_1 z + \ldots + a_n z^n| \le |a_0| + |a_1||z| + \ldots + |a_n||z|^n$

Hence, $M(r) = \max_{|z|=r} |p(z)| \le |a_0| + |a_1|r + \ldots + |a_n|r^n$

$\le r^n(|a_0| + \ldots + |a_n|)$ (choosing $r \ge 1$. Since ultimately $r \to \infty$, the choice is justified).

$= Br^n$, where $B = |a_0| + \ldots + |a_n|$. Hence

$\log M(r) \le \log B + n \log r \le \log r + n \log r$ (Taking r sufficiently large).

$= (n + 1) \log r$. Now,

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} \le \limsup_{r\to\infty} \frac{\log(n+1) + \log\log r}{\log r} = 0$$

i.e. $\rho \le 0$. But by definition $\rho \ge 0$. Hence $\rho = 0$

(ii) Here $M(r) = e^{|k|r}$ and hence

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = \limsup_{r\to\infty} \frac{\log(|k|r)}{\log r} = 1$$

(iii) We know that

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

and so

$$|\sin z| \le |z| + \frac{|z|^3}{3!} + \frac{|z|^5}{5!} + \cdots = r + \frac{r^3}{3!} + \frac{r^5}{5!} + \cdots = \sinh r \text{ on } |z| \le r.$$

$= \dfrac{e^r - e^{-r}}{2}$. Also at $z = ir$, $\sin z = \dfrac{e^{-r} - e^r}{2i}$ and so $|\sin z| = \dfrac{e^r - e^{-r}}{2}$.

Hence $M(r) = \dfrac{e^r - e^{-r}}{2} = \dfrac{e^r(1 - e^{-2r})}{2}$

$$\log M(r) = r + \log\left(\frac{1 - e^{-2r}}{2}\right) = r\left\{1 + \frac{1}{r}\log\left(\frac{1 - e^{-2r}}{2}\right)\right\}$$

Therefore,

$$\lim_{r\to\infty} \frac{\log\log M(r)}{\log r} = \lim_{r\to\infty}\left[1 + \log\left\{1 + \frac{1}{r}\log\left(\frac{1 - e^{-2r}}{2}\right)\right\}/\log r\right] = 1$$

So order of sin z is 1.

(iv) Following as in (iii) we find that the order of $\cos\sqrt{z}$ = 1/2.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. We now state a theorem which will give us order of f(z) in terms of the coefficients a_n of the power series expansion of f(z).

Theorem : Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of finite order ρ. Then,

$$\rho = \limsup_{n\to\infty} \frac{-\log n}{\log|a_n|^{1/n}} = \limsup_{n\to\infty} \frac{-n\log n}{\log|a_n|}$$

6.8 The function n(r)

Let f(z) be an entire function with zeros at the points a_1, a_2, ..., arranged in order of non-decreasing modulus, i.e. $|a_1| \le |a_2| \le \cdots$, multiple zeros being repeated according to their multiplicities. We define the function $n(r)$ to be the number of zeros of $f(z)$ in $|z| \le r$. Evidently $n(r)$ is a non-decreasing, non-negative function of r which is constant in any interval which does not contain the modulus of a zero of $f(z)$. We observe that if $f(0) \ne 0$. $n(r) = 0$ for $r < |a_1|$. Also, $n(r) = n$ for $|a_n| \le r < |a_{n+1}|$.

Jensen's inequality can also be written in the following form involving $n(r)$.

Theorem 6.12 (Jensen's inequality) : Let $f(z)$ be an entire function with $f(0) \ne 0$, and $a_1, a_2, \ldots$ be the zeros of $f(z)$ such that $|a_1| \le |a_2| \le \cdots$, multiple zeros being repeated according to their multiplicities. If $|a_N| \le r < |a_{N+1}|$, then

$$\log \frac{r^N}{|a_1 \cdots\cdots a_N|} = \int_0^r \frac{n(x)}{x}\,dx \le \log M(r) - \log|f(0)| \tag{113}$$

Proof : Let $|a_i| = r_i$, $i = 1, 2, \ldots$, and r be a positive number such that $r_N \le r < r_{N+1}$. Let $x_1 \ldots, x_m$ be the distinct numbers of the set $A = \{r_1, \ldots, r_N\}$ where $x_1 = r_1, \ldots, x_m = r_N$. Suppose x_i is repeated p_i times in A. Then, $p_1 + \ldots + p_m = N$. Also let $t_i = p_1 + \ldots + p_i$, $i = 1, \ldots, m$.

We now consider two cases.

Case 1) Let $r_N < r$. Then,

$$\int_0^r \frac{n(x)}{x}dx = \lim_{\varepsilon\to 0}\left\{\int_{x_1}^{x_2-\varepsilon} \frac{n(x)}{x}dx + \int_{x_2}^{x_3-\varepsilon} \frac{n(x)}{x}dx + \ldots + \int_{x_{m-1}}^{x_m-\varepsilon} \frac{n(x)}{x}dx\right\} + \int_{x_m}^{r} \frac{n(x)}{x}dx$$

(since $\int_0^{x_1-\varepsilon} \frac{n(x)}{x}dx = 0$ as $n(x) = 0$ for $0 \le x < x_1$).

$$= \lim_{\varepsilon\to 0}\left\{\int_{x_1}^{x_2-\varepsilon} \frac{t_1}{x}dx + \int_{x_2}^{x_3-\varepsilon} \frac{t_2}{x}dx + \cdots + \int_{x_{m-1}}^{x_m-\varepsilon} \frac{t_{m-1}}{x}dx\right\} + \int_{r_N}^{t} \frac{N}{x}dx$$

$$= \lim_{\varepsilon\to 0}\{[t_1 \log x\}_{x_1}^{x_2-\varepsilon} + [t_2 \log x]_{x_1}^{x_1-\varepsilon} + \cdots + [t_{m-1}\log x]_{x_{m-1}}^{x_m-\varepsilon} + [N \log x]_{r_N}^{r}$$

$$= \lim_{\varepsilon\to 0}[t_1\{\log(x_2-\varepsilon) - \log x_1\} + t_2\{\log(x_3-\varepsilon) - \log x_2\} +$$

$$\cdots + t_{m-1}\{\log(x_m-\varepsilon) - \log x_{m-1}\}] + N(\log r - \log r_N)$$

$= t_1 (\log x_2 - \log x_1) + t_2 (\log x_3 - \log x_2) + \ldots$

$+ t_{m-1} (\log x_m - \log x_{m-1}) + N(\log r - \log r_N)$

$= p_1 \log x_2 - p_1 \log x_1 + (p_1 + p_2) \log x_1 - (p_1 + p_2) \log x_2 + \ldots + (p_1 + \ldots + p_{m-1})$

$\log x_m - (p_1 + \ldots + p_{m-1}) \log x_{m-1} + N \log r - (p_1 + \ldots + p_m) \log x_m$

$= N \log r - (p_1 \log x_1 + p_2 \log x_2 + \ldots + p_m \log x_m)$

$$= \log r^N - \log x_1^{p_1} x_2^{p_2} \cdots x_m^{p_m} = \log \frac{r^N}{x_1^{p_1} x_2^{p_2} \cdots x_m^{p_m}}$$

$$= \log \frac{r^N}{r_1 \cdots r_N}$$ Thus,

$$\int_0^r \frac{n(x)}{x} dx = \log \frac{r^N}{|a_1 \cdots a_N|} \qquad (114)$$

Case 2). Let $r_N = r$. As before,

$$\int_0^r \frac{n(x)}{x} dx = \lim_{\varepsilon \to 0} \left\{ \int_{x_1}^{x_2 - \varepsilon} \frac{t_1}{x} dx + \cdots + \int_{x_{m-1}}^{x_m - \varepsilon} \frac{t_{m-1}}{x} dx \right\}$$

$$= \sum_{i=1}^{m-1} t_i (\log x_{i+1} - \log x_i) + t_m (\log r - \log r_N)$$

$$= \log \frac{r^N}{|a_1 \cdots a_N|}$$ (Proceeding as in case 1).

Thus in any case,

$$\int_0^r \frac{n(x)}{x} dx = \log \frac{r^N}{|a_1 \cdots a_N|} \cdot$$ But Jensen's inequality gives us

$$\frac{r^N}{|a_1 \cdots a_N|} \le \frac{M(r)}{|f(0)|} \cdot$$ Hence,

$$\int_0^r \frac{n(x)}{x} dx = \log \frac{r^N}{|a_1 \cdots a_N|} \le \log M(r) - \log |f(0)|.$$

Theorem 6.13 : If f(z) be an entire function with finite order ρ, then $n(r) = O(r^{\rho + \varepsilon})$ for ε > 0 and for sufficiently large values of r.

Proof : By Jensen's inequalilty,

$$\int_0^r \frac{n(x)}{x} dx \le \log M(r) - \log |f(0)| \qquad (115)$$

We replace r by 2r in (115) and obtain

$$\int_0^{2r} \frac{n(x)}{x} dx \le \log M(2r) - \log |f(0)| \qquad (116)$$

Since order of f(z) is ρ we have for any ε > 0,

$\log M(2r) < (2r)^{\rho + \varepsilon} = Kr^{\rho + \varepsilon}$ for all large r, K being a constant. Hence from (116).

$\int_0^{2r} \frac{n(x)}{x} dx < Ar^{\rho+\varepsilon}$ for all large r, A being a constant independent of r. Since n(x) is non-negative and non-decreaing function of x, $\int_r^{2r} \frac{n(x)}{x} dx \le \int_0^{2r} \frac{n(x)}{x} dx <$ $Ar^{\rho+\varepsilon}$ and also $\int_r^{2r} \frac{n(x)}{x} dx \ge \int_r^{2r} \frac{n(r)}{x} dx = n(r)\log 2$

Hence, $n(r)\log 2 \le \int_r^{2r} \frac{n(x)}{x} dx < Ar^{\rho+\varepsilon}$,

i.e., $n(r) < \frac{A}{\log 2} r^{\rho+\varepsilon}$ for all large r. Hence, $n(r) = O(r^{\rho+\varepsilon})$.

6.9 Convergence exponent (or, exponent of Convergence)

Let f(z) be an entire function with zeros at the points a_1, a_2, ..., arranged in order of non-decreasing modulus, multiple zeros being repeated according to their multiplicities and $|a_i| = r_i$, i = 1, 2, ..., We define convergence exponent ρ_1 of the zeros of f(z) by the equation

$$\rho_1 = \limsup_{n\to\infty} \frac{\log n}{\log r_n} \tag{117}$$

or, equivalently by $\rho_1 = \limsup_{n\to\infty} \frac{\log n(r)}{\log r}$ (118)

The convergence exponent has the following property.

Theorem 6.14 : Let f(z) be an entire function with zeros at a_1 a_2, ..., arranged in order of non-decreasing modulus, multiple zeros being repeated according to their multiplicities and $|a_i| = r_i$. If the convergence exponent ρ_1 of the zeros of f(z) is finite, then the series $\sum_{n=1}^{\infty} \frac{1}{r_n^{\alpha}}$ converges when $\alpha > \rho_1$ and diverges when $\alpha < \rho_1$.

If ρ_1 is infinite, the above series diverges for all positive values of α.

Proof : Let ρ_1 be finite and $\alpha > \rho_1$. Then, $\rho_1 < \frac{1}{2}(\rho_1 + \alpha)$.

Hence, $\frac{\log n}{\log r_n} < \frac{1}{2}(\rho_1 + \alpha)$ for all large n.

or, $\log n < \log r_n^{\frac{1}{2}(\rho_1+\alpha)}$, i.e.

$n < r_n^{\frac{1}{2}(\rho_1+\alpha)}$; or, $\frac{2}{n^{\rho_1+\alpha}} < r_n$ i.e.,

$$r_n^{\alpha} > \frac{2\alpha}{n^{\rho_1+\alpha}} = n^{1+\frac{\alpha-\rho_1}{\alpha+\rho_1}} = n^{1+p},\text{ where } p = \frac{\alpha-\rho_1}{\alpha+\rho_1} > 0.$$

Hence, $\frac{1}{r_n^{\alpha}} < \frac{1}{n^{1+p}}$ for all large n. Hence,

$\sum_{n=1}^{\infty} \frac{1}{r_n^{\alpha}}$ converges.

Next, let $\alpha < \rho_1$. Then, $\frac{\log n}{\log r_n} > \alpha$ for a sequence of values of n, tending to infinity. That is, $\log n > \log r_n^{\alpha}$

$$\text{or, } \frac{1}{r_n^{\alpha}} > \frac{1}{n} \qquad (119)$$

for a sequence of values of n tending to infinity.

Let N be such a value of n for which (119) holds and m be the least integer > $\frac{N}{2}$. Then, as r_n is non-decreasing,

$$\sum_{n=N-m}^{N} \frac{1}{r_n^{\alpha}} = \frac{1}{r_{N-m}^{\alpha}} + \frac{1}{r_{N-m+1}^{\alpha}} + \cdots + \frac{1}{r_N^{\alpha}} \geq \frac{1}{r_N^{\alpha}} + \cdots + \frac{1}{r_N^{\alpha}}$$

$= \frac{m+1}{r_N^{\alpha}} > \frac{m}{r_N^{\alpha}} > \frac{m}{N} > \frac{1}{2}$. Since N may be as large as we please, by Cauchy's principle of convergence, the series $\sum_{n=1}^{\infty} \frac{1}{r_n^{\alpha}}$ diverges.

If ρ_1 is infinite, then for any positive value of α, $\frac{\log n}{\log r_n} > \alpha$ for a sequence of values of n tending to infinity; i.e., $n > r_n^{\alpha}$ for a sequence of values of n tending to infinity. Hence as before, the series $\sum_{n=1}^{\infty} \frac{1}{r_n^{\alpha}}$ diverges for any positive α.

Note 1. Observe that ρ_1 may also be defined as the lower bound of the positive numbers α for which the series $\sum_{n=1}^{\infty} \frac{1}{r_n^{\alpha}}$ is convergent. If f(z) has no zeros we define $\rho_1 = 0$ and if $\sum_{n=1}^{\infty} \frac{1}{r_n^{\alpha}}$ diverges for all positive α, then $\rho_1 = \infty$.

Note 2. If ρ_1 is finite, the series $\sum_{n=1}^{\infty} \frac{1}{r_n^{\rho_1}}$ may be convergent or divergent. For example, if $r_n = n$, then $\rho_1 = \limsup_{n\to\infty} \frac{\log n}{\log r_n} = 1$

and $\sum_{n=1}^{\infty} \frac{1}{r_n^{\rho_1}} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Again, if $r_n = n(\log n)^2$,

then, $\rho_1 = \limsup_{n\to\infty} \frac{\log n}{\log n + 2\log\log n} = 1$, and

$$\sum_{n=1}^{\infty} \frac{1}{r_n^{\rho_1}} = \sum_{n=1}^{\infty} \frac{1}{n(\log n)^2}$$ converges.

Theorem 6.15 : If f(z) is an entire function with finite order ρ and $r_1, r_2, ...,$ are the moduli of the zeros of f(z),

then $\sum_{n=1}^{\infty} \frac{1}{r_n^{\alpha}}$ converges if $\alpha > \rho$.

Proof : We choose β such that $\rho < \beta < \alpha$. Since for any $\varepsilon > 0$,

$$n(r) = 0\,(r^{\rho+\varepsilon}),\ n(r) < Kr^{\beta} \qquad (120)$$

for all large r, K being a constant.

Putting $r = r_n$, n large, (120) gives $n < Kr_n^{\beta}$, i.e.,

$r_n > \frac{n^{1/\beta}}{k^{1/\beta}}$ or, $\frac{1}{r_n^{\alpha}} < \frac{B}{n^{\alpha/\beta}}$ for all large n, B being a constant. Since $\frac{\alpha}{\beta} > 1$, $\sum_{n=1}^{\infty} \frac{1}{r_n^{\alpha}}$ converges.

Corollary : Since convergence exponent ρ_1 is the lower bound of positive numbers α for which $\sum_{n=1}^{\infty} \frac{1}{r_n^{\alpha}}$ is convergent, it follows that $\rho_1 \le \rho$.

Note : ρ_1 may be 0 or ∞. For example if $r_n = e^n$, $\rho_1 = 0$ and if $r_n = \log n$, then $\rho_1 = \infty$. For the function $f(z) = e^z$, $\rho = 1$ and $\rho_1 = 0$ so that $\rho_1 < \rho$. But for sin z or cos z, $\rho = \rho_1 = 1$.

Result : If the convergence exponent ρ_1 of the zeros of an entire function f(z) is greater than 0, then f(z) has infinite number of zeros.

Proof : If possible, suppose f(z) has finite number of zeros with moduli $r_1, \ldots, r_N$. The series $\sum_{n=1}^{N} \frac{1}{r_n^{\alpha}}$, being of finite number of terms, converges for every $\alpha > 0$. Hence $\rho_1 = 0$, a contradiction. Hence f(z) has infinite number of zeros.

Note : For an entire function with finite number of zeros, $\rho_1 = 0$.

Example : Find the convergence exponent of the zeros of cos z.

Solution : First method : The zeros of cos z are $\frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}, \ldots$

Now, $\sum_{n=1}^{\infty} \frac{1}{r_n^{\alpha}} = \left(\frac{2}{\pi}\right)^{\alpha} + \left(\frac{2}{\pi}\right)^{\alpha} + \left(\frac{2}{\pi}\right)^{\alpha} \cdot \frac{1}{3^{\alpha}} + \cdots$

$= 2\left(\frac{2}{\pi}\right)^{\alpha}\left(1 + \frac{1}{3^{\alpha}} + \frac{1}{5^{\alpha}} + \cdots\right)$. The series $\frac{1}{1^{\alpha}} + \frac{1}{3^{\alpha}} + \frac{1}{5^{\alpha}} + \cdots$

converges when $\alpha > 1$ and diverges when $\alpha < 1$. Hence the lower bound of the positive numbers α for which $\sum_{n=1}^{\infty} \frac{1}{r_n^{\alpha}}$ converges is 1 i.e., $\rho_1 = 1$.

Second method : The zeros of cos z are $(2n+1)\frac{\pi}{2}$,

$n = 0, \pm 1, \pm 2, \cdots$; i.e. $\frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}, \cdots$

Let $a_1 = \frac{\pi}{2}$, $a_1' = -\frac{\pi}{2}$, $a_2 = \frac{3\pi}{2}$, $a_2' = -\frac{3\pi}{2}$, $\cdots$,

$a_n = (2n-1)\frac{\pi}{2}$, $a_n' = -(2n-1)\frac{\pi}{2}$, $\cdots$, Hence,

$r_1 = |a_1| = |a_1'| = \frac{\pi}{2}$, $r_2 = |a_2| = |a_2'| = \frac{3\pi}{2}, \cdots$, $r_n = |a_n| = |a_n'| =$

$(2n-1)\frac{\pi}{2}, \cdots$ Hence, $\rho_1 = \limsup_{n\to\infty} \frac{\log n}{\log r_n}$

$$= \limsup_{n\to\infty} \frac{\log n}{\log(2n-1) + \log\frac{\pi}{2}} = \limsup_{n\to\infty} \frac{\log n}{\log\left\{n\left(2-\frac{1}{n}\right)\right\} + \log\frac{\pi}{2}}$$

$$= \limsup_{n\to\infty} \frac{1}{1 + \frac{\log\left(2-\frac{1}{n}\right)}{\log n} + \frac{\log \pi / 2}{\log n}} = 1.$$

6.10 Canonical Product

Let f(z) be an entire function with infinite number of zeros at a_n, n = 1, 2, ... $a_n \neq 0$. If there exists a least non-negative integer p such that the series $\sum_{n=1}^{\infty} \frac{1}{r_n^{p+1}}$ is convergent, where $r_n = |a_n|$, we form the infinite product $G(z) = \prod_{n=1}^{\infty} E\left(\frac{z}{a_n}, p\right)$. By Weirstrass' factor theorem G(z) represents an entire function having zeros precisely at the points a_n. We call G(z) as the Canonical product corresponding to the sequence $\{a_n\}$ and the integer p is called its genus. If z = 0 is a zero of f(z) of order m, then the canonical product is $z^m G(z)$.

Observe that if the convergence exponent $\rho_1 \neq$ an integer, then $p = [\rho_1]$ and if $\rho_1 =$ an integer, then $p = \rho_1$ when $\sum_{n=1}^{\infty} \frac{1}{r_n^{\rho_1}}$ is divergent and $p = \rho_1 - 1$ if $\sum_{n=1}^{\infty} \frac{1}{r_n^{\rho_1}}$ is convergent.

In any case, $\rho_1 - 1 \leq p \leq \rho_1 \leq \rho$, where ρ = order of f(z).

Examples : (i) Let $a_n = n$. Then $\sum_{n=1}^{\infty} \frac{1}{r_n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent while $\sum_{n=1}^{\infty} \frac{1}{r_n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. So, p = 1.

(ii) Let $a_n = e^n$. Then p = 0.

We now state an important theorem without proof. The proof can be found in any standard book.

Borel's theorem : The order of a canonical product is equal to the convergence exponent of its zeros.

Example : Find the canonical product of $f(z) = \sin z$.

Solution : $f(z)$ is an entire function with infinite number of zeros at $z = n\pi$, n being an integer. First we consider the zeros of $f(z)$ excluding the simple zero at $z = 0$. Let $a_n = n\pi$, $n = \pm 1, \pm 2, \ldots$

$|a_n| = r_n$. Then, $r_n = |n\pi|$. Now, $\sum_{n=1}^{\infty} \frac{1}{r_n} = \sum_{n=1}^{\infty} \frac{1}{|n\pi|}$

$= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, but $\sum_{n=1}^{\infty} \frac{1}{r_n^2} = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. Hence genus of the required canonical product $p = 1$.

Hence the canonical product G(z) is given by

$$G(z) = \prod_{n=-\infty}^{\infty} E\left(\frac{z}{a_n}, 1\right), \text{ where } \prod_{n=-\infty}^{\infty}{}' \text{ means } n = 0 \text{ is excluded in the product.}$$

$$= \prod_{n=-\infty}^{\infty}{}' \left(1 - \frac{z}{n\pi}\right) e^{\frac{z}{n\pi}} = \prod_{n=1}^{\infty}{}' \left\{\left(1 - \frac{z}{n\pi}\right) e^{\frac{z}{n\pi}} \cdot \left(1 - \frac{z}{n\pi}\right) e^{-\frac{z}{n\pi}}\right\}$$

$= \prod_{n=-\infty}^{\infty}{}' \left(1 - \frac{z^2}{n^2\pi^2}\right)$. Since origin is a simple zero of sin z, the required canonical product of sin z is given by

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right).$$

Exercises

1. Find the order of the entire functions :

   (a) sinh z (b) $e^z \sin z$, (c) e^{z^n}, (d) e^{e^z}, (e) cos z, (f) $e^{p(z)}$, where $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_n \neq 0$, (g) $\sum_{n=0}^{\infty} \frac{z^n}{(n!)^{\alpha}}, \alpha > 0$, (h) $\sum_{n=0}^{\infty} \left(\frac{e\alpha}{n}\right)^{n/\alpha} z^n, \alpha > 0$

2. Given $f_1(z)$ and $f_2(z)$ are two entire functions of orders ρ_1 and ρ_2 respectively, show that (i) order of $f_1(z) f_2(z)$ is $\leq \max(\rho_1, \rho_2)$ (ii) order of $f_1(z) + f_2(z)$ is $\leq \max(\rho_1, \rho_2)$, and equality occurs if $\rho_1 \neq \rho_2$.

3. Find the convergence exponent of the zeros of sin z.

4. Find the canonical product of cos z.

5. Show that if a > 1, the entire function $\prod_{n=1}^{\infty}\left(1-\frac{z}{n^a}\right)$ is of order $\frac{1}{a}$.

6.11 Hadamard's Factorization Theorem

Before taking up Hadamard's factorization theorem we state a theorem due to Borel and Caratheodory.

Borel and Caratheodory's theorem : Let f(z) be analytic in

$|z| \le R, M(r) = \max_{|z|=r}|f(z)|,\ A(r) = \max_{|z|=r}\{\operatorname{Re} f(z)\}.$

Then for 0 < r < R,

$$M(r) \le \frac{2r}{R-r}A(R) + \frac{R+r}{R-r}|f(0)| < \frac{R+r}{R-r}\{A(R)+|f(0)|\} \tag{121}$$

Proof : Omitted (cf. Theory of entire functions–A.S.B Holland- p. 53).

Corollary : $\max_{|z|=r}\left|f^{(n)}(z)\right| \le \frac{2^{n+2}.n!R}{(R-r)^{n+1}}\left(A(R)+|f(0)|\right)$ (122)

Hadamard's Factorization Theorem 6.16 :

If f(z) is an entire function of finite order ρ with infinite number of zeros and f(0) ≠ 0, then f(z) = $e^{Q(z)}$ G(z), where G(z) is the canonical product formed with the zeros of f(z) and Q(z) is a polynomial of degree not greater than ρ.

Proof : By Weierstrass' factor theorem we already have

$f(z) = e^{Q(z)}G(z)$ (123)

where G(z) is the canonical product with genus p formed with the zeros a_1, a_2, ... of f(z) and Q(z) is an entire function. Since ρ is finite we need to show that Q(z) is a polynomial of degree ≤ ρ. Let m = [ρ]. Then, p ≤ m. Taking logarithms on both sides of (123) we have,

$$\log f(z) = Q(z) + \log G(z)$$

$$= Q(z) + \sum_{n=1}^{\infty}\log\left(1-\frac{z}{a_n}\right) + \sum_{n=1}^{\infty}\left\{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+...+\frac{1}{p}\left(\frac{z}{a_n}\right)^p\right\} \tag{124}$$

Differentiating both sides of (124) m + 1 times,

$$\frac{d^m}{dz^m}\left(\frac{f^1(z)}{f(z)}\right) = Q^{(m+1)}(z) - m!\sum_{n=1}^{\infty}\frac{1}{(a_n - z)^{m+1}} \tag{125}$$

[Since $p \le m, \frac{d^{m+1}}{dz^{m+1}}\sum_{n=1}^{\infty}\left\{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + ... + \frac{1}{p}\left(\frac{z}{a_n}\right)^p\right\} = 0$

and $\frac{d^{m+1}}{dz^{m+1}}\log\left(1 - \frac{z}{a_n}\right) = \frac{d^{m+1}}{dz^{m+1}}\log(a_n - z) = -m!\frac{1}{(a_n - z)^{m+1}}$]

Now, Q(z) will be a polynomial of degree m at most if we can show that $Q^{(m+1)}(z) = 0$.

Let $g_R(z) = \frac{f(z)}{f(0)}\prod_{|a_n| \le R}\left(1 - \frac{z}{a_n}\right)^{-1}$. Then $g_R(z)$ is an entire function and $g_R(z) \neq 0$ in $|z| \le R$. [Since f(z) is entire, $f(0) \neq 0$ and $\prod_{|a_n| \le R}\left(1 - \frac{z}{a_n}\right)^{-1}$ cancels with factors in f(z)].

For $|z| = 2R$ and $|a_n| \le R$ we have, $\left|1 - \frac{z}{a_n}\right| \ge 1$. Hence,

$$|g_R(z)| \le \frac{|f(z)|}{|f(0)|} < Ae^{(2R)^{\rho+\varepsilon}} \text{ for } |z| = 2R \tag{126}$$

By maximum modulus theorem, $|g_R(z)| < Ae^{(2R)^{\rho+\varepsilon}}$ (127)

for $|z| < 2R$. Let $h_R(z) = \log g_R(z)$ such that $h_R(0) = 0$.

Then $h_R(z)$ is analytic in $|z| \le R$. Hence from (127)

$\text{Re } h_R(z) = \log|g_R(z)| < KR^{\rho+\varepsilon}$, K = Constant (128)

Hence from the corollary of the theorem of Borel and Caratheodory we have

$$\left|h_R^{(m+1)}(z)\right| \le \frac{2^{m+3}(m+1)|!R}{(R-r)^{m+2}}.KR^{\rho+\varepsilon} \text{ for } |z| = r < R$$

Hence for $|z| = r = \frac{R}{2}$,

$$\left|h_R^{(m+1)}(z)\right| = 0\left(R^{\rho+\varepsilon-m-1}\right) \tag{129}$$

But $h_R(z) = \log g_R(z) = \log f(z) - \log f(0) - \sum_{|a_n| \le R} \log\left(1 - \frac{z}{a_n}\right)$

Hence $h_R^{(m+1)}(z) = \frac{d^m}{dz^m}\left(\frac{f'(z)}{f(z)}\right) + m! \sum_{|a_n| \le R} \frac{1}{(a_n - z)^{m+1}}$

$$= 0(R^{\rho+\varepsilon-m-1}) + 0\left(\sum_{|a_n|>R} \frac{1}{|a_n|^{m+1}}\right) \qquad (130)$$

for $|z| = \frac{R}{2}$ and so also for $|z| < \frac{R}{2}$ by maximum modulus theorem. The first term on the right of (130) tends to 0 as $R \to \infty$ if $\varepsilon > 0$ is small enough since $m + 1 > \rho$. Also the second term tends to 0 since $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{m+1}}$ is convergent.

In fact, $\sum_{|a_n|>R} \frac{1}{|a_n|^{m+1}}$ becomes the remainder term for large R.

Hence $Q^{(m+1)}(z) = 0$ since $Q^{(m+1)}(z)$ is independent of R.

Thus, Q(z) is a polynomial of degree not greater than ρ.

6.12 Consequences of Hadamard's Theorem

Theorem 6.17 : An entire function of finite order admits any finite complex number except, perhaps, one number.

Proof. Let us suppose that f does not admit two finite values a and b. Then $f(z) - a \neq 0$ for all z in $\mathbb{C}$ and hence there exists an entire function g(z) such that

$$f(z) - a = e^{g(z)}$$

The function f(z) – a is of finite order since f(z) has finite order. Following Hadamard's factorization theorem g(z) must be a polynomial. Now $e^{g(z)}$ does not assume the value b – a i.e. $g(z) \neq \log(b - a)$ for any z in $\mathbb{C}$. As because g(z) is a polynomial it contradicts the essence of the Fundamental Theorem of Algebra [(14), Th. 3.11, page-65].

Theorem 6.18 : An entire function of fractional order possesses infinitely many zeros.

Proof. Let f be an entire function of fractional order ρ. If possible, suppose the zeros of f(z) are $\{a_1, a_2, \ldots a_n)$, finite in number, counted according to their multiplicity. Then f(z) can be expressed as

$$f(z) = e^{g(z)} (z - a_1)(z - a_2) \dots (z - a_n)$$

where g(z) is an entire function. Applying Hadamard's factorization theorem, the degree of the polynomial $g(z) \leq \rho$. It is easy to check that f(z) and $e^{g(z)}$ are of same order. But we have already seen that the order of $e^{g(z)}$ is exactly the degree of g(z), which is an integer. This implies ρ is an integer. This contradiction completes the proof.

6.13 Meromorphic Functions

The term meromorphic comes from the **Ancient Greek** "meros" meaning part, as opposed to "holos" meaning whole. This function is analytic on a domain D except a set of isolated points, which are poles for the function.

Definition : A function f(z) analytic in a domain D except for poles is said to be meromorphic.

Theorem 6.19 : A rational function is meromorphic.

Proof : Let f(z) = p(z)/q(z) where p and q are polynomials with no common zeros. If the degree of p is less than or equal to the degree of q, then f has only a finite number of poles and the point at infinity is not a pole. On the otherhand, if the degree of p is greater than the degree of q, then (taking degree of p(z) = m and degree of q(z) = n).

$$f(z) = \frac{a_m z^m + a_{m-1} z^{m-1} + \dots + a_1 z + a_0}{b_n z^n + b_{n-1} z^{n-1} + \dots + b_1 z + b_0}$$

$$= c_{m-n} z^{m-n} + c_{m-n-1} z^{m-n-1} + \dots + c_1 z + c_0 + \frac{r(z)}{q(z)}$$

where degree of $r(z) \leq n - 1$. This shows that the point at infinity is a pole of order (m – n) and there lie a finite number of poles in the unextended plane. These establish that f(z) is meromorphic.

Theorem 6.20 : [Partial fraction decomposition]. Let p(z), q(z) be two polynomials with no common zeros and that $0 \leq \deg(p) < \deg(q)$. Let $a_1, \dots a_k$ be the zeros of q(z) with multiplicities $\alpha_1, \dots, \alpha_k$. Then p(z)/q(z) can be expressed uniquely as

$$\frac{p(z)}{q(z)} = \sum_{i=1}^{k} \sum_{j=1}^{\alpha_i} \frac{c_{ij}}{(z - a_i)^j} \qquad (131)$$

Proof. The decomposition is unique. We assume that the relation (131) exists. Let r > 0 be small enough. Then for $z \in N(a_i, r)$, (131) can be rewritten as

$$\frac{p(z)}{q(z)} = g(z) + \sum_{j=1}^{\alpha_i} \frac{c_{ij}}{(z-a_i)^j} \tag{132}$$

since $N(a_i, r)$ does not contain any zero of q(z) other than a_i, g(z) is analytic at $z = a_i$.

Multiplying both sides of (132) by $(z - a_i)^{\alpha_i}$, we obtain

$$\frac{p(z)}{q(z)}(z-a_i)^{\alpha_i} = g(z)(z-a_i)^{\alpha_i} + \sum_{j=1}^{\alpha_i} c_{ij}(z-a_i)^{\alpha_i - j} \tag{133}$$

Now the function $\frac{p(z)}{q(z)}(z-a_i)^{\alpha_i}$ is analytic for all z belonging to $N(a_i, r)$ and hence can be expanded in a Taylor series in a neighbourhood of a_i in $N(a_i, r)$

$$\frac{p(z)}{q(z)}(z-a_i)^{\alpha_i} = \sum_{n=0}^{\infty} c_n (z-a_i)^n \tag{134}$$

Combining (133) and (134), we write

$$\sum_{n=0}^{\alpha} c_n (z-a_i)^n = g(z)(z-a_i)^{\alpha_i} + c_{i\alpha_i} + c_{i\alpha_{i-1}}(z-a_i) + \ldots + c_{i1}(z-a_i)^{\alpha_i - 1}$$

Comparing the coefficients we find

$c_i\alpha_i = c_0, c_{i\alpha_i - 1} = c_1, \ldots, c_{i1} = c_{\alpha_i - 1}$ uniquely

Existence of the decomposition.

The principal part associated to each pole a_i is

$$\sum_{j=1}^{\alpha_i} \frac{c_{ij}}{(z-a_i)^j}$$

Now if we subtract all the principal parts we find the function

$$f(z) = \frac{p(z)}{q(z)} - \sum_{i=1}^{k} \sum_{j=1}^{\alpha_i} \frac{c_{ij}}{(z-a_i)^j}$$

is analytic in the extended plane. Now each of the terms

$$\frac{c_{ij}}{(z-a_i)^j}$$

converges to zero for $z \to \infty$, and also p(z)/q(z) converges to zero for $z \to \infty$ since deg(q) > deg(p). This shows that f(z) $\to$ 0 for $z \to \infty$. But then f is necessarily bounded and hence constant by Liouville's theorem. A constant function tending to zero as $z \to \infty$ must be identically zero.

Example 4 : Consider the rational function

$$\frac{p(z)}{q(z)} = \frac{2z^3 + (5i+3)z^2 + (3-5i)}{z^4 - 1}$$

We can write this as

$$\frac{p(z)}{q(z)} = \frac{\alpha}{z-1} + \frac{\beta}{z+1} + \frac{\gamma}{z-i} + \frac{\delta}{z+i} \tag{135}$$

$$= g_1(z) + \frac{\alpha}{z-1}$$

considering z belonging to $|z - 1| < 1$. Then

$$\frac{p(z)}{q(z)}(z-1) = g_1(z)(z-1) + \alpha \Rightarrow \alpha = 2$$

6.14 Partial Fraction Expansion of Meromorphic Functions

Let f(z) be a meromorphic function and z_0 be a pole of order m with the principal part

$$p(z) = \frac{c_{-m}}{(z-z_0)} + \frac{c_{-m+1}}{(z-z_0)^{m+1}} + \ldots + \frac{c_{-1}}{z-z_0}$$

Then f(z) can be written as [see § 6.2, (14)]

$$f(z) = p(z) + g(z)$$

where g(z) is an entire function. Now if, in general, $z_1, z_2, \ldots, z_n$ are the poles of a meromorphic function f with the corresponding principl parts $P_1, P_2 \ldots, P_n$ then f can be expressed as

$$f(z) = \sum_{j=1}^{n} P_j(z) + \psi(z) \tag{136}$$

where $\psi(z)$ is an entire function.

But the question arises whether it is possible to construct a meromorphic function possessing poles at the sequence of points $\{z_n\}$ with corresponding principal parts $P_1, P_2 \ldots$ Because in this case the series $\Sigma P_j(z)$ in (136) turns out to be an infinite series $\sum_{j=1}^{n} P_j(z)$, which needs to be convergent.

Gösta Mittag Leffler (1846-1927), German in origin but his several generations lived in Sweden, overcame this difficulty by introducing a polynomial $p_n(z)$ dependent on z_n and $P_n(z)$ so that the series $\sum_{n=1}^{\infty}\{P_n(z)-p_n(z)\}$ is uniformly convergent in any compact set K not containing any points of the sequence $\{z_n\}$.

Theorem 6.21 [The Mittag Leffler Theorem] : Given a sequence of distinct complex numbers $\{z_n\}$,

$$|z_1| \le |z_2| \le \ldots, \lim_{n\to\infty} z_n = \infty$$

and a sequence of rational functions $\{P_n(z)\}$,

$$P_n(z) = \sum_{k=1}^{ln} \frac{c_{nk}}{(z-z_n)^k},\ 1_n \ge 1,\ n = 1,\ 2,\ldots \tag{137}$$

there exists a meromorphic function f(z) having poles at the points z_n and only there with $P_n(z)$ as its principal part at z_n and can be represented in the form of an expansion

$$f(z) = \sum_{n=1}^{\infty}[P_n(z)-p_n(z)] + h(z)$$

where h(z) is an arbitrary entire function and $p_n(z)$ is suitable partial sum of Taylor's expansion of the singular part which is analytic in the open disc $|z| < |z_n|$.

Proof. Without loss of generality we assume that z = 0 is not a pole of f(z). Now $P_k(z)$ is analytic for $|z| < |z_k|$ and can be expanded in this neighbourhood of z :

$$P_k(z) = \sum_{j=0}^{\infty} c_j^{(k)} z^j$$

and hence this series converges uniformly in the disk $|z| \le |z_k|/2$. Let $p_k(z) = \sum_{j=0}^{\alpha_k} c_j^{(k)} z^j$ be a partial sum of this expansion such that

$$|P_k(z)-p_k(z)| < \frac{1}{k^2} \text{ for } |z| \le |z_k|/2.$$

Let R be an arbitrary large positive number and since $z_n \to \infty$ as $n \to \infty$ we can find an N(R) so large that $|z_n| > 2R$ when $n \ge N(R)$. Therefore in the circle $|z| < R < \frac{|z_N|}{2}$

$$\sum_{n=1}^{\infty}\left[P_n(z)-p_n(z)\right] = \sum_{n=1}^{N(R)-1}\left[P_n(z)-p_n(z)\right] + \sum_{n=N(R)}^{\infty}\left[P_n(z)-p_n(z)\right]$$

the first sum in the r.h.s is finite and the second sum $\sum_{N(R)}^{\infty}$ is absolutely and uniformly convergent by comparison with the convergent series $\sum_{n=N(R)}^{\infty} 2^{-n}$. Therefore $\sum_{n=1}^{\infty}[P_n(z) - p_n(z)]$ is analytic in $|z| < R$ except at the poles belonging to the sequence $\{z_n\}$. It is thus a meromorphic function with the poles at z_1, z_2, ... and with the principal parts $P_1(z)$, $P_2(z)$, ... at each point z_n respectively. Now if f(z) possesses the same poles only with the same principal parts then

$$f(z) - \sum_{n=1}^{\infty}[P_n(z) - p_n(z)]$$

is an entire function h(z), say. This completes the proof.

Example 5 : Prove that

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=-\infty}^{\infty}{}' \left\{\frac{1}{z-n} + \frac{1}{n}\right\}$$

Solution : The given function $\pi \cot \pi z$ has simple poles at $z = 0, \pm 1, \pm 2, \ldots$ with residue 1.

Here,

$$\frac{1}{z-n} = -\frac{1}{n}\frac{1}{\left(1-\frac{z}{n}\right)} = -\frac{1}{n}\left(1 + \frac{z}{n} + \frac{z^2}{n^2} + \ldots\right), |z| < n \tag{138}$$

Let $|z| < R$ and N(R) be so large that $R < \frac{n}{2}$ when $n \geq N(R)$. Then from (138), we find

$$\left|\frac{1}{z-n} + \frac{1}{n}\right| \leq \frac{2R}{N^2}, \ n \geq N$$

Now, since $\Sigma 1/N^2$ is convergent, the series

$$\sum_{n=-\infty}^{\infty}{}' \left\{\frac{1}{z-n} + \frac{1}{n}\right\}$$

converges uniformly on any compact set (lying in $|z| < R$) not containing any of the points $z = \pm 1, \pm 2, \ldots$ Therefore applying the Mittag-Leffler theorem we can express

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=-\infty}^{\infty}{}' \left\{\frac{1}{z-n} + \frac{1}{n}\right\} + h(z) \tag{139}$$

where h(z) is an entire function. Differentiating term-wise, we obtain

$$\pi^2 \operatorname{cosec}^2 \pi z = \frac{1}{z^2} + \sum_{n=-\infty}^{\infty}{}' \frac{1}{(z-n)^2} - h'(z)$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} - h'(z)$$

and $$h'(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} - \pi^2 \operatorname{cosec}^2 \pi z = f(z) - \psi(z), \text{say} \tag{140}$$

We notice that the functions f(z) and ψ(z) are both periodic with period 1 and consequently h′(z) is also periodic with the same period.

Let z = x + iy. Consider the strip $0 \le x \le 1$. In fact, the convergence of the series in (140) is uniform for $y \ge 1$, say and the limit tends to 0 as $y \to \infty$ (this can be seen on taking the limit in each term of the series).

Again, sin(x + iy) = sin x cos (iy) + cos x sin (iy)

= sin x cosh y + i cos x sinh y

and so

$$|\sin \pi z|^2 = |\sin \pi(x+iy)|^2$$

$$= \sin^2 \pi x \cosh^2 \pi y + \cos^2 \pi x \sinh^2 \pi y$$

$$= \cosh^2 \pi y - \cos^2 \pi x$$

which establishes that $\pi^2 \operatorname{cosec}^2 \pi z$ tends uniformly to zero as $y \to \infty$. From these we conclude that h′(z) is bounded in the period strip $0 \le x \le 1$ and due to its periodicity it is bounded in the entire plane. By Liouville's theorem it then reduces to a constant. Now since

$$\lim_{y\to\infty} h'(z) = \lim_{y\to\infty} f(z) - \lim_{y\to\infty} \psi(z) = 0 - 0 = 0$$

h′(z) is indeed zero and h(z) = c, a constant. Then from (139),

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=-\infty}^{\infty}{}' \left(\frac{1}{z-n} + \frac{1}{n} \right) + c$$

For, $z = \frac{1}{2}$

$$0 = 2 + \sum_{1}^{\infty} \left(\frac{2}{1-2k} + \frac{2}{1+2k} \right) + c$$

$$= 2 + 2\left\{\left(\frac{1}{-1} + \frac{1}{3}\right) + \left(-\frac{1}{3} + \frac{1}{5}\right) + \left(-\frac{1}{5} + \frac{1}{7}\right) + \ldots\right\} + c$$

$$= 2 - 2 + c$$

$\Rightarrow$ c = 0 i.e. $h(z) \equiv 0$. Finally we obtain

$$\pi \cot \pi z = \frac{1}{z} + \sum_{z=-\infty}^{\infty}{}' \left\{\frac{1}{z-n} + \frac{1}{n}\right\}$$

Now since the series on the r.h.s is uniformly convergent on any compact set not containing the points z = 0, ±1, ±2 ..., rearrangement of the terms are permissible and hence

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \tag{141}$$

Remark : Here it is proved incidentally that

$$\pi^2 \text{cosec}^2 \pi z = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \tag{142}$$

[see equation (140)]

We can now utilize the identity (141) to calculate easily some familiar sums. Here the l.h.s of (141) has the Laurent series expansion in the neighbourhood of z = 0.

$$\pi \cot \pi z = \frac{1}{z} - \frac{\pi^2 z}{3} - \frac{\pi^4 z^3}{45} - \frac{2\pi^6 z^5}{945} - \ldots$$

Note that the series on the r.h.s of (141) converges uniformly near z = 0. By Th. 4.14 [14] it converges uniformly together with all derivatives. Again

$$\frac{2z}{z^2 - n^2} = -2\left(\frac{z}{n^2} + \frac{z^3}{n^4} + \frac{z^5}{n^6} + \ldots\right)$$

and we obtain easily,

$$\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} \tag{143}$$

Example 6. Prove that

$$\pi \tan \pi z = -\sum_{n=-\infty}^{\infty} \left[\frac{1}{z - \left(n + \frac{1}{2}\right)} + \frac{1}{n + \frac{1}{2}}\right]$$

[or, equivalently, $\pi \tan \pi z = 2z\sum_{n=0}^{\infty}\left[\left(n+\frac{1}{2}\right)^2 - z^2\right]^{-1}$]

Solution : Here the given function π tan πz possesses simple poles at $z = \pm\frac{1}{2},\ \pm\frac{3}{2},\ \ldots$ with residue –1.

Then, $$\frac{-1}{z-\left(n+\frac{1}{2}\right)} = \frac{1}{\left(n+\frac{1}{2}\right)\left(1-\frac{z}{n+\frac{1}{2}}\right)} = \frac{1}{n+\frac{1}{2}}\left[1+\frac{z}{n+\frac{1}{2}}+\left(\frac{z}{n+\frac{1}{2}}\right)^2+\cdots\right]$$

and the series

$$\sum_{n=-\infty}^{\infty}\left[\frac{-1}{z-\left(n+\frac{1}{2}\right)} - \frac{1}{n+\frac{1}{2}}\right]$$

converges uniformly on any compact set not containing any of the poles of the given function. By Mittag-Leffler theorem,

$$\pi \tan \pi z = -\sum_{n=-\infty}^{\infty}\left[\frac{1}{z-\left(n+\frac{1}{2}\right)} + \frac{1}{n+\frac{1}{2}}\right] + h(z)$$

where h(z) is an arbitray entire function. Now proceeding as in example 5, we can have the desired result.

Example 7 : Establish that

$$\frac{1}{e^z - 1} = -\frac{1}{2} + \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2 + 4n^2\pi^2}$$

Solution : We rewrite $1/e^z - 1$ as

$$\frac{1}{e^z - 1} = \frac{e^{-z/2}}{e^{z/2} - e^{-z/2}} = \frac{1}{2}\frac{e^{-z/2} - e^{z/2} + e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}} = -\frac{1}{2} + \frac{1}{2}\coth\frac{z}{2}$$

But $\coth\frac{z}{2} = \dfrac{\cosh\frac{z}{2}}{\sinh\frac{z}{2}} = \dfrac{i\cos\left(i\frac{z}{2}\right)}{\sin\left(i\frac{z}{2}\right)} = i\cot\left(i\frac{z}{2}\right)$

Now utilising (141) we get the result.

6.15 Partial Fraction Expansion of Meromorphic Functions Using Residue theorem

Let us suppose f to be a meromorphic function whose only singularities are simple poles $z_1, z_2, \ldots$ with increasing moduli $0 < |z_1| \le |z_2| \le \ldots,$

$\lim_{n\to\infty} z_n = \infty$ and Res $(f(z); z_n) = A_n$. Suppose there exists a sequence $\{C_n\}$ of simple closed contours such that

(i) C_n does not contain any of the poles z_k

(ii) each C_n lies inside C_{n+1}

(iii) $\min_{z\in C_n}|z| = R_n \to +\infty$ as $n \to +\infty$

(iv) length of C_n is $0(R_n)$

(v) $\max_{z\in C_n}|f(z)| = 0(R_n)$

Then
$$f(z) = f(0) + \sum_{k=1}^{\infty} A_k\left(\frac{1}{z - z_k} + \frac{1}{z_k}\right) \tag{144}$$

The series (144) converges uniformly in any bounded domain not containing the poles of f(z).

To prove the above result we consider the integral

$$I_n(z) = \frac{1}{2\pi i}\int_{C_n}\frac{zf(\varsigma)}{\varsigma(\varsigma - z)}\,d\varsigma \tag{145}$$

where $z \in$ Int C_n and $z \ne z_k$ $(k = 1, 2, \ldots)$

Here the integrand in (145) possesses simple poles at $\varsigma = 0$, $\varsigma = z$ and $\varsigma = z_k \in$ Int C_n. Then using the Residue theorem, we find from (145) that

$$I_n(z) = \left[\frac{zf(\varsigma)}{\varsigma - z}\right]_{\varsigma=0} + \left[\frac{zf(\varsigma)}{\varsigma}\right]_{\varsigma=z} + \left[\frac{1}{\varsigma(\varsigma - z)}\right]_{\varsigma=z_k} \operatorname{Res}(f(\varsigma); z_k)$$

$$= -f(0) + f(z) + \sum_{z_k \in \text{Int}C_n} \frac{zA_k}{z_k(z_k - z)}$$

Thus,

$$f(z) = f(0) + \sum_{z_k \in \text{Int}C_n} A_k\left(\frac{1}{z - z_k} + \frac{1}{z_k}\right) + \frac{1}{2\pi i}\int_{C_n} \frac{zf(\varsigma)}{\varsigma(\varsigma - z)} d\varsigma \qquad (146)$$

We now show that $\lim_{n\to\infty} |I_n(z)| = 0$ for $|z| < R$.

$$|I_n(z)| \le \frac{|z|}{2\pi}\int_{C_n} \frac{|f(\varsigma)|}{|\varsigma||\varsigma - z|}|d\varsigma| < \frac{R}{2\pi}\int_{C_n} \frac{|f(\varsigma)|}{|\varsigma||\varsigma - R|}|d\varsigma| \to 0$$

as $n \to \infty$ by the given conditions (iii), (iv) and (v).

Then (144) follows from (146) considering all the contours C_1, C_2, ... etc.

Example 8 : If α_n are positive roots of the equation tan z = z, show that

$$\frac{z \sin z}{\sin z - z\cos z} = \frac{3}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - \alpha_n^2}$$

where $\left(n - \frac{1}{2}\right)\pi < \alpha_n < \left(n + \frac{1}{2}\right)\pi.$

Solution : Given α_n are positive roots of tan z = z, so $\pm\, \alpha_n$ are roots of sin z – z cos z = 0. To check whether the function f(z)/g(z), where f(z) = z sin z and g(z) = sin z – z cos z, has any pole at z = 0 we notice that

$f'(z) = \sin z + z\cos z$	$g'(z) = z\sin z = f(z)$
$f''(z) = 2\cos z - z\sin z$	$g''(z) = f'(z)$
$f'(0) = 0$ but $f''(0) \neq 0$	$f''(z) = g'''(z)$
	so, $g'(0) = g''(0) = 0$ but $g'''(0) \neq 0$

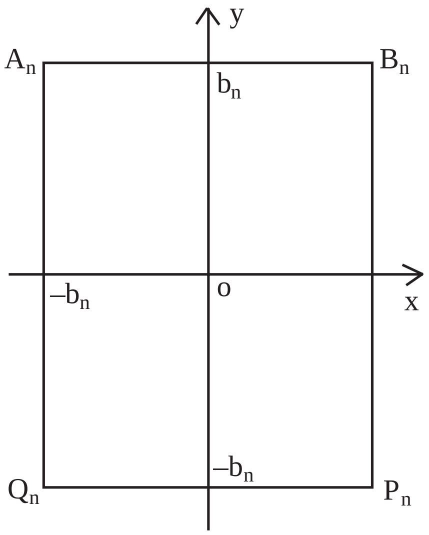
Thus origin is the double zero of f(z) and triple zero of g(z). As a result the given function f/g possesses a simple pole at z = 0. To find its residue at z = 0 we note that

$$\frac{f''(z)}{(z^2)''} = 1 \text{ and } \frac{g'''(z)}{(z^3)'''} = \frac{1}{3}$$

and so residue there is 3. Thus the function $F(z) = \frac{z\sin z}{\sin z - z\cos z} - \frac{3}{z}$ has the simple poles at $z = \pm\alpha_n$ as its only singularities and Res $(F(z); \pm\alpha_n) = 1$ and $F(0) = 0$ since $F(z) = -F(-z)$.

Since $\left(n-\frac{1}{2}\right)\pi < \alpha_n < \left(n+\frac{1}{2}\right)\pi$, we consider the sequence of contours $\{C_n\}$, formed by the straight lines $x = \pm b_n$, $y = \pm b_n$ with $b_n = \left(n+\frac{1}{2}\right)\pi$, n = 1, 2...,

$A_nB_nP_nQ_n$ shown below :



We find that when $z \in B_nP_n$, $z = b_n + iy$, where $-b_n \le y \le b_n$.

Hence,

$$|\cot z| = \left|\frac{\cos\left\{\left(n+\frac{1}{2}\right)\pi + iy\right\}}{\sin\left\{\left(n+\frac{1}{2}\right)\pi + iy\right\}}\right|$$

$$= \left|\frac{\sin(iy)}{\cos(iy)}\right| = \left|\frac{e^y - e^{-y}}{e^y + e^{-y}}\right| \qquad (147)$$

Same result holds when $z \in A_nQ_n$. Now when z lies on either of the lines A_nB_n or Q_nP_n, $z = x \pm i\left(n+\frac{1}{2}\right)\pi$

$$|\cot z| = \left|\frac{\cos\left\{x \pm i\left(n+\frac{1}{2}\right)\pi\right\}}{\sin\left\{x \pm i\left(n+\frac{1}{2}\right)\pi\right\}}\right| \ge \frac{\sinh\left(n+\frac{1}{2}\right)\pi}{\cosh\left(n+\frac{1}{2}\right)\pi}$$

$$= \frac{1-e^{-2(n+1)^{\pi}}}{1+e^{-(2n+1)^{\pi}}} \ge \frac{e^{\pi}-1}{e^{\pi}+1} \qquad (148)$$

The given function can be rewritten as

$$\frac{z\sin z}{\sin z - z\cos z} = \frac{1}{\frac{1}{z} - \cot z}$$

I. Bound on the sides A_nQ_n & B_nP_n of the square C_n : Using (147), we obtain

$$\left|\frac{1}{\frac{1}{z}-\cot z}\right| \le \frac{1}{|\cot z|-\frac{1}{|z|}} = \frac{1}{\left|\frac{e^y-e^{-y}}{e^y+e^{-y}}\right|-\frac{1}{\sqrt{b_n^2+y^2}}} \to 1 \text{ as } n \to \infty.$$

II. Bound on the sides A_nB_n & Q_nP_n of C_n : Here we apply (148) to achieve

$$\left|\frac{1}{\frac{1}{z}-\cot z}\right| \le \frac{1}{|\cot z|-\frac{1}{|z|}} \le \frac{1}{\frac{e^\pi-1}{e^\pi+1}-\frac{1}{\sqrt{b_n^2+y^2}}} \to \frac{e^\pi+1}{e^\pi-1} \text{ as } n \to \infty.$$

Thus,

$$\left|\frac{z\sin z}{\sin z - z\cos z}\right| \le \frac{e^\pi+1}{e^\pi-1},\ z \in C_n,\ n = 1, 2, \ldots$$

This shows that the function F(z) is bounded on the sequence of contours $\{C_n\}$ and we can apply (144) to prove

$$\frac{z\sin z}{\sin z - z\cos z} = \frac{3}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{z-\alpha_n}+\frac{1}{\alpha_n}+\frac{1}{z+\alpha_n}-\frac{1}{\alpha_n}\right]$$

$$= \frac{3}{2} + \sum_{n=1}^{\infty}\frac{2z}{z^2-\alpha_n^2}$$

Exercises

1. Obtain partial fraction expansion of cosec z.
2. Prove that

$$\sec z = \sum_{n=1}^{\infty}(-1)^n \frac{(2n-1)\pi}{z^2-\left(n-\frac{1}{2}\right)^2\pi^2}$$

3. Show that

$$\tan z = -\sum_{n=1}^{\infty}\frac{2z}{z^2-\left(n-\frac{1}{2}\right)^2\pi^2}$$

and hence deduce

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$$

6.16 The Gamma Function

The gamma function Γ(z) was introduced by Swedish Mathematician L. Euler (1707-1783), in 1729 while he was seeking for a function of a real variable x which is continuous for positive x and reduces to x! when x is a positive integer. Gamma function is widely used in the fields of probability and statistics, as well as combinatorics.

Gamma function Γ(z) can be introduced in either of the ways :

(i) in terms of infinite product

(ii) in the form of infinite integral

(iii) in limit formula

We establish the form (i) first considering the fact that it possesses simple poles at z = 0, –1, –2, ... and nowhere vanishes in the entire plane and satisfies

$$z\Gamma(z) = \Gamma(z+1),\ \Gamma(1) = 1 \tag{149}$$

To construct Γ(z) we claim that f(z) = 1/Γ(z) is entire with simple zeros at z = –n (n = 0, 1, 2, ...).

Again we know that k = 1 is the largest non-negative integer for which

$$\sum_{n=1}^{\infty}\frac{1}{n^k}$$

diverges. Then utilizing the Weierstrass Factorization theorem f(z) can be represented as

$$f(z) = ze^{g(z)}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{\frac{-z}{n}}$$

where g(z) is an entire function, so that gamma function will be of the form

$$\Gamma(z) = e^{-g(z)}\frac{1}{z\prod_{1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}} \tag{150}$$

Now we find g(z) so that (149) hold. We write (150) in the form

$$\Gamma(z) = \lim_{n\to\infty} \frac{e^{-g(z)}}{z\prod_{1}^{n}\left(1+\frac{z}{m}\right)e^{\frac{-z}{m}}}$$

$$= \lim_{n\to\infty} \frac{n!\exp\left[-g(z)+\sum_{1}^{n}\frac{z}{m}\right]}{z(z+1)\ldots\ldots(z+n)} = \lim_{n\to\infty}\Gamma_n(z), \text{ say} \qquad (151)$$

$$\frac{z\Gamma_n(z)}{\Gamma_n(z+1)} = \frac{n!\,z\exp\left[-g(z)+\sum_{1}^{n}\frac{z}{m}\right]}{z(z+1)\ldots\ldots(z+n)}\,\frac{(z+1)(z+2)\ldots\ldots(z+n+1)}{n!\exp\left[-g(z+1)+\sum_{1}^{n}\frac{z+1}{m}\right]}$$

$$= (z+n+1)\exp\left[g(z+1)-g(z)-\sum_{1}^{n}\frac{1}{m}\right]$$

$$= \left(1+\frac{z+1}{n}\right)n\exp\left[g(z+1)-g(z)-\sum_{1}^{n}\frac{1}{m}\right]$$

$$= \left(1+\frac{z+1}{n}\right)\exp\left[g(z+1)-g(z)-\sum_{1}^{n}\frac{1}{m}+\log n\right]$$

Now from the relation $\frac{z\Gamma(z)}{\Gamma(z+1)} = \lim_{n\to\infty}\frac{z\Gamma_n(z)}{\Gamma_n(z+1)}$, we find that

$$\frac{z\Gamma(z)}{\Gamma(z+1)} = \lim_{n\to\infty}\left(1+\frac{z+1}{n}\right)\exp\left[g(z+1)-g(z)-\sum_{1}^{n}\frac{1}{m}+\log n\right]$$

$$= \exp[g(z+1)-g(z)-\gamma]$$

where
$$\gamma = \lim_{n\to\infty}\left(\sum_{1}^{n}\frac{1}{m}-\log n\right) = 0\cdot 57722 \qquad (152)$$

is known as the Euler's constant.

Thus in order that the conditions in (149) to hold, we should have

$$g(z+1) - g(z) = \gamma + 2k\pi i \ (k \equiv \text{integer}) \qquad (153)$$

and

$$1 = \Gamma(1) = \lim_{n\to\infty} \Gamma_n(1) = \lim_{n\to\infty} \frac{e^{-g(1)+\sum_1^n \frac{z}{m} - \log n}}{1+\frac{1}{n}} = e^{-g(1)+\gamma}$$

so that $$g(1) = \gamma + 2j\pi i \quad (j \equiv \text{integer}) \tag{154}$$

The simplest entire function satisfying (154) is given by

$$g(z) = \gamma z$$

Finally from (150),

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_1^{\infty} \left(1+\frac{z}{n}\right)^{-1} e^{z/n} \tag{155}$$

Gauss's Formula

From (151) we have the representation

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\exp\left[\left(\sum_1^n \frac{1}{m} - \gamma\right)z\right]}{z(z+1)\ldots\ldots(z+n)}$$

$$= \lim_{n\to\infty} \frac{n!\exp\left[\left\{\left(\sum_1^n \frac{1}{m} - \gamma - \log n\right) + \log n\right\}z\right]}{z(z+1)\ldots\ldots(z+n)}$$

$$= \lim_{n\to\infty} \frac{n!\,n^z}{z(z+1)\ldots\ldots(z+n)}, \text{ since } \lim_{n\to\infty}\left(\sum_1^n \frac{1}{m} - \log n - \gamma\right) = 0 \tag{156}$$

The above expression for $\Gamma(z)$, $z \neq 0, -1, -2, \ldots$ is termed as Gauss's formula, though it was first derived by Euler.

In many places it is known as Euler's limit formula.

Example 9 : Let

$$\Gamma(z, n) = \frac{n!\,n^z}{z(z+1)\ldots\ldots(z+n)}$$

Prove that

$$\Gamma(z, n) = \frac{n^z \Gamma(n+1)\Gamma(z)}{\Gamma(n+z+1)}$$

and hence deduce that

$$\frac{n^z\Gamma(n)}{\Gamma(n+z)} \to 1 \text{ as } n \to \infty$$

Solution :

$$\Gamma(n + z + 1) = z(z + 1)(z + 2)......(z + n)\ \Gamma(z)$$

so, $$\frac{n^z\Gamma(n+1)\Gamma(z)}{\Gamma(n+z+1)} = \frac{n^z\Gamma(n+1)}{z(z+1)(z+2)\ldots\ldots(z+n)} = \frac{n!\,n^z}{z(z+1)(z+2)\ldots\ldots(z+n)} = \Gamma(z,n)$$

Now,

$$\frac{n^z\Gamma(n)}{\Gamma(n+z)} = \frac{(n+z)\Gamma(z,n)}{n\Gamma(z)}$$

$$\lim_{n\to\infty}\frac{n^z\Gamma(n)}{\Gamma(n+z)} = \lim_{n\to\infty}\left(1+\frac{z}{n}\right)\frac{\lim_{n\to\infty}\Gamma(z,n)}{\Gamma(z)} = 1 \text{ by Gauss's formula.}$$

In the expression (155) for Γ(z) the infinite product is uniformly convergent on every compact subset of $\mathbb{C}$ – {0, –1, ...}. So calculating Γ′(z)/Γ(z) we find that

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\left(-\frac{1}{n+z}+\frac{1}{n}\right)$$

This function $\frac{\Gamma'(z)}{\Gamma(z)}$ is denoted by ψ(z) and named as Gaussian psi function and it is seen from its expression that ψ is meromorphic in $\mathbb{C}$ with simple poles at z = 0, –1, –2, ... and Res(ψ; –n) = –1 for n = 0, 1, 2, ...

Example 10 : Show that

(i) ψ(1) = –γ

(ii) $\psi(z+1) - \psi(z) = \frac{1}{z}$

(iii) $\psi(z) - \psi(1-z) = -\pi \cot \pi z$.

Solution :

(i) $$\psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\left(-\frac{1}{n+z}+\frac{1}{n}\right)$$

so, $$\psi(1) = -\gamma - 1 + \sum_{n=1}^{\infty}\left(-\frac{1}{n+1} + \frac{1}{n}\right)$$

$$= -\gamma - 1 + \left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \cdots\right)$$

$$= -\gamma.$$

(ii) $$\psi(z+1) - \psi(z) = -\gamma - \frac{1}{z+1} + \sum_{n=1}^{\infty}\left(-\frac{1}{n+z+1} + \frac{1}{n}\right) - \sum_{n=1}^{\infty}\left(-\frac{1}{n+z} + \frac{1}{n}\right) + \gamma + \frac{1}{z}$$

$$= \frac{1}{z} - \frac{1}{z+1} + \sum_{n=-1}^{\infty}\left(\frac{1}{n+z} - \frac{1}{n+z+1}\right)$$

$$= \frac{1}{z} - \frac{1}{z+1} + \left(\frac{1}{z+1} - \frac{1}{z+2} + \frac{1}{z+2} - \frac{1}{z+3} + \cdots\right)$$

$$= \frac{1}{z}.$$

(iii) $$\psi(z) - \psi(1-z) = -\frac{1}{z} + \frac{1}{1-z} + \sum_{1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+z}\right) - \sum_{1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1-z}\right)$$

$$= -\frac{1}{z} - \frac{1}{z-1} + \sum_{1}^{\infty}\left(\frac{1}{n+1-z} - \frac{1}{n+z}\right)$$

$$= -\frac{1}{z} - \frac{1}{z-1} - \frac{1}{z+1} - \frac{1}{z-2} - \frac{1}{z+2} - \cdots$$

$$= -\frac{1}{z} - \left(\frac{1}{z-1} + \frac{1}{z+1}\right) - \left(\frac{1}{z-2} + \frac{1}{z+2}\right) - \cdots$$

$$= -\frac{1}{z} - \sum_{1}^{\infty}\frac{2z}{z^2 - n^2} = -\pi\cot\pi z, \text{ by (141)}$$

6.17 A Few Properties of $\Gamma(z)$

We have $$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{1}^{\infty}\left(1 + \frac{z}{n}\right)e^{-z/n}$$

Hence, $$\frac{1}{\Gamma(z)\Gamma(-z)} = -z^2 \prod_{1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

$$= -\frac{z}{\pi} \pi z \prod_{1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

$$= -\frac{z}{\pi} \sin \pi z$$

or, $$\frac{1}{\Gamma(z)[-z\Gamma(-z)]} = \frac{\sin \pi z}{\pi}$$

i.e. $$\frac{1}{\Gamma(z)\Gamma(1-z)} = \frac{\sin \pi z}{\pi},$$ [using $z\Gamma(z) = \Gamma(z + 1)$ i.e., $-z\Gamma(-z) = \Gamma(1 - z)$] (157)

In particular, $\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ (minus sign is excluded since $\Gamma\left(\frac{1}{2}\right)$ is positive by (155)). Likewise using

$$\Gamma(z + 1) = z\Gamma(z)$$

we find

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}.\frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

and in general

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{1.3.\ldots\ldots(2n-1)}{2^n}\sqrt{\pi}, \ (n = 1, 2, \cdots$$

i.e. $$\Gamma\left(n + \frac{1}{2}\right) / \sqrt{\pi} = \frac{(2n)!}{n!(2)^{2n}} \qquad (158)$$

If n is a positive integer repeated use of (149) produce

$$\Gamma(n+1) = n!$$

The Γ-function can therefore be considered as an extension of the factorial function to the complex plane.

Legendre's Duplication Formula

Let us consider the Gauss's formula

$$\Gamma(z)=\lim_{n\to\infty}\frac{n!n^z}{z(z+1)\ldots\ldots(z+n)}=\lim_{n\to\infty}\Gamma(z,n),\ \text{say}$$

Then,

$$\Gamma(2z,2n)=\frac{(2n)!(2n)^{2z}}{2z(2z+1)\ldots(2z+n)\ldots\ldots(2n+2z)}$$

$$=\frac{2^{2n}n!\Gamma\left(n+\frac{1}{2}\right)\left(\sqrt{\pi}\right)^{-1}(2n)^{2z}}{2z(2z+1)(2z+2)\ldots\ldots(2z+2n)}\ \text{[Replacing (2n)! by (158)]}$$

$$=\frac{2^{2z-1}n!(n)^{2z}\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}z(z+1)(z+2)\ldots(z+n)\left(z+\frac{1}{2}\right)\left(z+\frac{3}{2}\right)\ldots\left(z+n-\frac{1}{2}\right)}$$

$$=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z,n)\Gamma\left(n+\frac{1}{2}\right)\frac{1}{\left(z+\frac{1}{2}\right)\left(z+\frac{3}{2}\right)\ldots\left(z+n-\frac{1}{2}\right)}$$

$$=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z,n)\Gamma\left(n+\frac{1}{2}\right)\frac{\Gamma\left(z+\frac{1}{2},n\right)}{n^{1/2}\Gamma(n)}\frac{z+\frac{1}{2}+n}{n}$$

and $\Gamma(2z)=\lim_{n\to\infty}\Gamma(2z,2n)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)\lim_{n\to\infty}\left[\frac{\Gamma\left(n+\frac{1}{2}\right)}{n^{1/2}\Gamma(n)}\frac{z+\frac{1}{2}+n}{n}\right]$

$$=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)\ \text{[using example 9]}$$

So that

$$\sqrt{\pi}\Gamma(2z)=2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)\qquad(159)$$

This is known as Legendre's duplication formula.

Residue of $\Gamma(z)$ at its poles

$\Gamma(z)$ is analytic throughout the complex plane except at its only singularities which are simple poles situated at z = 0, –1, –2, ... That is $\Gamma(z)$ is analytic in the right half of the complex plane Re z > 0. Using the fact that $z\Gamma(z) = \Gamma(z + 1)$, we have

$\Gamma(z+n+1) = (z+n)(z+n-1)(z+n-2)\ldots(z+1)z\Gamma(z)$, $n \equiv$ positive integer and

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\ldots\ldots(z+n-1)(z+n)}$$

$$\text{Res }(\Gamma(z); -n) = \lim_{z\to -n}(z+n)\Gamma(z)$$

$$= \lim_{z\to -n}\frac{\Gamma(z+n+1)}{z(z+1)\ldots.(z+n-1)}$$

$$= \frac{(-1)^n}{n!}, \; n = 0,1,2,\ldots$$

Integral representation of $\Gamma(z)$

Theorem : Prove that

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt \text{ for Re } z > 0.$$

Proof. Let

$$F_n(z) = \frac{n!n^z}{z(z+1)\ldots.(z+n)}$$

We prove the theorem in the following two steps :

(i) $F_n(z) = \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}dt$

(ii) $\lim_{n\to\infty}\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}dt = \int_0^\infty e^{-t}t^{z-1}dt$

To establish (i) we change the variable t to ns in

$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}dt$$

to obtain

$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}dt = n^z\int_0^1 (1-s)^n s^{z-1}\,ds$$

Now integrating by parts we find the right hand side is equal to

$$n^z\left[\frac{1}{z}s^z(1-s)^n\Big|_0^1 + \frac{n}{z}\int_0^1 (1-s)^{n-1}s^z ds\right]$$

$$= n^z\,\frac{n}{z}\int_0^1 (1-s)^{n-1}s^z ds$$

$$= n^z\,\frac{n.(n-1)....1}{z(z+1)....(z+n-1)}\int_0^1 s^{z+n-1}ds \text{ [Integrating by parts (n – 1) times]}$$

$$= \frac{n!n^2}{z(z+1)....(z+n)} = F_n(z)$$

Now to prove (ii)we show that

$$\lim_{n\to\infty}\int_0^n \left[e^{-t} - \left(1-\frac{t}{n}\right)^n\right]t^{z-1}dt = 0, \text{ Re } z > 0 \qquad (161)$$

For this, note that

$$1+\frac{t}{n} \le e^{\frac{t}{n}} \le \frac{1}{1-\frac{t}{n}} \text{ for } |t| < n \qquad (162)$$

Then, $\left(1+\frac{t}{n}\right)^n \le e^t$ and $\left(1-\frac{t}{n}\right)^n \le e^{-t}$;

Consequently,

$$0 \le e^{-t} - \left(1-\frac{t}{n}\right)^n = e^{-t}\left[1-e^t\left(1-\frac{t}{n}\right)^n\right] \le e^{-t}\left[1-\left(1-\frac{t^2}{n^2}\right)^n\right]$$

$$= e^{-t}\,\frac{t^2}{n^2}\left[1+\left(1-\frac{t^2}{n^2}\right)+\ldots+\left(1-\frac{t^2}{n^2}\right)^{n-1}\right] \le e^{-t}\,\frac{t^2}{n}.$$

Therefore,

$$\left|\int_0^n \left[e^{-t} - \left(1-\frac{t}{n}\right)^n\right]t^{z-1}dt\right| < \frac{1}{n}\int_0^n e^{-t}t^{\text{Re } z+1}dt$$

which approaches zero as $n \to \infty$ because the integral on the right converges. This completes the proof of (ii). Finally combining the results (i) and (ii) with the Gauss's formula (156) we get

$$\Gamma(z) = \lim_{n\to\infty} F_n(z) = \lim_{n\to\infty} \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}dt = \int_0^\infty e^{-t}t^{z-1}dt$$

References

1. M. J. Ablowitz and A. S. Fokas, Complex variables, 2nd edition, Cambridge University Press, 2003.
2. L. V. Ahlfors, Complex analysis, 2nd edition, McGraw-Hill, 1979.
3. R. Buck, Advanced Calculus.
4. J. B. Conway, Functions of one complex variable, Springer International Student Edition, 1973.
5. E. T. Copson, An introduction to the theory of functions of a complex variable, Clarendon Press, Oxford, 1935.
6. J. W. Dettman, Applied complex variables, Macmillan, New York, 1965.
7. J. G. Krzyz, Problems in complex variable theory. American Elsevier Publishing Company, INC, New York, 1971.
8. W. K. Hayman, Multivalued functions, Cambridge University Press, 1958.
9. ...., Meromorphic functions, Clarendon Press, Oxford, 1964.
10. J. E. Marsden, Basic Complex analysis, W. H. Freeman and Company, Sanfrancisco, 1973.
11. A. I. Markushevich, Theory of functions of a complex variable-3 vols Prentice Hall, Engle Wood, Cliffs, N. J. 1965-67.
12. Z. Nehari, Conformal Mapping. McGraw-Hill, New York, 1952.
13. W. Rudin, Real and complex Analysis, Third Edition, McGraw-Hill, 1987.
14. P. K. Sengupta, Complex Analysis, Netaji Subhas Open University, 2003.
15. Y. V. Sidorov, M. V. Fedoryuk and M. I. Shabunin, Lectures on the theory of functions of a complex variable, Mir Publishers, Moscow, 1985.
16. E. C. Titchmarsh, The theory of functions, Oxford University Press, 1964.

NETAJI SUBHAS OPEN UNIVERSITY

STUDY MATERIAL

MATHEMATICS

POST GRADUATE

PG (MT) : IX A (II)
(Applied Mathematics)

Operations Research

PREFACE

In the curricular structure introduced by this University for students of Post-Graduate degree programme, the opportunity to pursue Post-Graduate course in Subjects introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation.

Keeping this in view, study materials of the Post-Graduate level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Co-operation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing and devising of a proper lay-out of the materials. Practically speaking, their role amounts to an involvement in invisible teaching. For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so that it may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University.

Needless to add, a great part of these efforts is still experimental-in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these do admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned.

Professor (Dr.) Subha Sankar Sarkar
Vice-Chancellor

Fifth Reprint : July, 2017

Printed in accrodance with the regulations and financial assistance of the Distance Education Bureau of the Uiversity Grants Commission.

Subject : Mathematics **Post Graduate**

Paper : PG (MT) : IX A (II)

Writer
Prof. T. K. Pal

Editor
Prof. R. N. Mukherjee

NETAJI SUBHAS OPEN UNIVERSITY

PG (MT): IX A (II)
Operations Research

Unit 1 ❑ Classical Optimization Techniques

Structure

1.2 Multivariable optimization with no constraints
1.1 Introduction
1.2 Multivariable optimization with no constraints

1.1 (Introduction)

The methods of determining relative extrema of functions of several variables using differential calculus are so old and well-known that they are referred to as classical. The classical methods of optimization are used in finding the optimum of continuous and differentiable functions. Since practical problems involve objective functions that are not continuous and/or differentiable, the classical optimization techniques have limited people of applications. But these classical techniques forms a basis for developing most of the numerical techniques of optimization.

In this unit we consider three types of problems viz

(i) Multivariable optimization with no constraints.

(ii) Multivariable optimization with equality constraints and

(iii) Multivariable optimization with inequality constraints

1.2 Multivariable optimization with no constraints

We develop the necessary and sufficient conditions for an n-variable functions f(x) to have extremt. It is assumed that the first and second partial derivatives of f(x) are continuous at every x.

Theorem 1.2.1 A necessary condition for x_0 to be an extreme point of $f(x)$ is that ∇ f(x_0) = 0 i.e. $\left[\frac{\partial t}{\partial x_i}\right]_{x0} = 0$ for i = 1, 2, ...n.

Proof : By Taylor's theorem we have

$$f(X_0 + h) = f(X_0) + \sum_{i=1}^{n} h_i\left[\frac{\partial t}{\partial x_i}\right]_{x0} + \frac{1}{\lfloor 2}\sum_{i=1}^{n}\sum_{j=1}^{n} h_i h_j\left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{x_0\theta h} \text{......(1)}$$

where $0 < \theta < 1$.

Since the last term is of order h_j^2, the terms of order h will dominate the higher order terms for small h. Thus the sign of $f(X_0 + h) - f(X_0)$ is decided by the sign of $\sum_{i=1}^{n} h_i\left[\frac{\partial t}{\partial x_i}\right]_{x0}$. Let X_0 be are extreme point, say miximum point. Then $f(X_0 + h) - f(X_0) > 0$ for all sufficiently small h. We are to show that $\left[\frac{\partial f}{\partial x_i}\right]_{x0} = 0 \ \forall i = 1, 2, \ldots\ldots, n$. If possible, let $\left[\frac{\partial f}{\partial x_k}\right]_{x0} \neq 0$.

Let us choose $h_i = 0$ for all $i \neq k$, and h_k sufficiently small. Then the sign of $f(X_0 + h) - f(X_0)$ is decided by the sign of $h_k\left[\frac{\partial f}{\partial x_k}\right]_{x0}$. Since $\left[\frac{\partial f}{\partial x_k}\right]_{x0} \neq = 0$, let $\left[\frac{\partial f}{\partial x_k}\right]_{x0} > 0$. Then $f(X_0 + h) - f(X_0)$ will be positive for $h_k > 0$ and negative for $h_k < 0$. This is a contradiction as x_0 is a minimum point. Similar contradiction occurs for $\left[\frac{\partial f}{\partial x_k}\right]_{x0} < 0$. Hence $\left[\frac{\partial f}{\partial x_k}\right]_{x0} \neq 0$ is not possible. $\therefore \left[\frac{\partial f}{\partial x_k}\right]_{x0} = 0$. This is true for any k = 1, 2,... n. Hence the theorem.

Theorem 1.2.2 A sufficient condition for a stationery point x_0 to be an extremum is that

(i) $\nabla f(X_0) = 0$ and the Hessian matrix $[H]_{x0}$ is positive definite when x_0 is a minumum point.

(ii) $\nabla f(X_0) = 0$ and the Hessian matrix $[H]_{x_0}$ is negative definite when x_0 is a maximum point.

Prob : By Tayloris theorem we have

$$f(X_0 + h) = f(X_0) + \sum_{i=1}^{n} h_i \left[\frac{\partial f}{\partial x_i}\right]_{x_0} + \frac{1}{\underline{|2}} \sum_{j=1}^{n} \sum_{j=1}^{n} h_i h_j \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{x_0 + oh}$$

Where $0 < \theta < 1$.

$f(X_0 + h) - f(X_0) = Q(x_0 + oh)$

Where $Q\ (x_0 + oh) = \frac{1}{\underline{|2}} \sum_{i=1}^{n} \sum_{j=1}^{n} h_i h_j \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{x_0 + oh}$

Now we have assumed that the second order partial derivative $\frac{\partial^2 f}{\partial x_i \partial x_j}$ is continuous in the neighbourhord of x_0. So far sufficiently small h, the signs of $Q(x_0 + oh)$ and $Q(x_0)$ are some. Hence $f(X_0 + h) - f(X_0)$ and $Q\ (x_0)$ have the same sign.

Let $J(X_0)$ be the Hessian matrix $\left[\left.\frac{\partial f}{\partial x_i \partial x_j}\right|_{x_0}\right]$. From matrix algebra we know that

$Q(X_0) = \frac{1}{\underline{|2}} \sum_{j=1}^{n} \sum_{j=1}^{n} h_i h_j \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{x_0}$ will be positive (negative) for al h if and only if the Hessian matrix $J(X_0)$ is positive definite (negative definite) at $X = X_0$.

Thus for sufficiently small h, the sign of $f(X_0 + h) - f(X_0)$ is positive (negative) if $J(X_0)$ is positive definite (negative definite) i.e., X_0 is a relative minimum (maximum) if $J\ (X_0)$ is positive definite (negative definite). Hence the theorem.

Result : Let $A = [a_{ij}]_{n \times n}$ and

$$A_1 = a_{11},\ A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{13} & a_{14} \end{vmatrix},\ A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Then the matrix A is

(i) positive definite iff $A_i > 0$ for all i = 1,2, ...n

(ii) negative definite iff the sign of Ai is (–i)i for i = 1,2, ...n.

(iii) positive semidefinite iff $A_i \geq 0$ for all i = 1, 2, ...n wih equality holding for at least one i

(iv) negative semidefinite iff $Ai \leq o$ for all i = 1,2, ... n with equality holding for at least one i

(v) indefinite if it is neither definite nor semidefinite.

Example 1.2.1 Determine the extreme points of the function

$$f(x_1, x_2) = x_1^3 + x_2^3 + 4x_x^2 + 2x_2^2 + 12$$

Solution :

$$\text{Here } \frac{\partial f}{\partial x_1} = 3x_1^2 + 8x_1, \ \frac{\partial f}{\partial x_2} = 3x_2^2 + 4x_2$$

The necessary condition for the existence of an extreme points gives

$$x_1 \ (3x_1 + 8) = 0 \text{ and } x_2 \ (3x_2 + 4) = 0$$

The solutions are (0, 0) (0, –4/3), (–8/3, 0< (–8/3, –4/3). The Hessian matrix of $f(x_1, x_2)$ is given by

$$J\ (x_1, x_2) = \begin{vmatrix} \frac{\partial^2 f}{dx_1^2} & \frac{\partial^2 f}{dx_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix} = \begin{vmatrix} 6x_1 + 8 & 0 \\ 0 & 6x_2 + 4 \end{vmatrix}$$

$$\therefore \ J_1 = 6x_1 + 8$$

$$\text{and } J_2 = \begin{vmatrix} 6x_1 + 8 & 0 \\ 0 & 6x_2 + 4 \end{vmatrix} = (6x_1 + 8)\ (6x_2 + 4)$$

For the point, (0, 0) we have

$J_1 = 6.0 + 8 = 8 > 0$ and $J_2 = (6.0 + 8)\ (6.0 + 4) = 32 > 0$

∴ J is positive definite. Hence (0, 0) is a relative minimum point of $f(x_1, x_2)$

For the point (0, –4/3) we have

$J_1 = 6.0 + 8 = 8 > 0$ and $J_2 = (6.0 + 8)(-6.4/3 + 4) = -32 < 0$

∴ J is indefinite. Hence (0, –4.3) is a saddle point of $f(x_1, x_2)$.

For the point (–8/3, 0) we have

$J_1 = -6.8/3 + 8 = -8 < 0$ and $J_2 = (-6.8/3 + 8)(6.0 + 4) = -32 < 0$.

∴ J is indefinite. Hence (–8/3, 0) is a paddle point of $f(x_1, x_2)$.

For the point (–8/3, –4/3) we have

$J_1 = -6.8/3 + 8 = -8 < 0$ and $J_2 = (-6.8/3 + 8)(-6.4/3 + 4) = 32 > 0$

∴ J is negative definite. Hence (–8/3, –4/3) is a relative maximum point of $f(x_1, x_2)$.

1.3 Multivariable optimization with equality constraints

We shall consider two methods viz

(i) Method of constrained variation and

(ii) Method of Lagrange multipliers.

The general multivariable optimization problem with equality constraints is

Minimize $f = f(X)$

subject to $g_i(X) = 0,\ i = 1, 2, \ldots\ldots, m$

Where $X = [x_1, x_2, \ldots\ldots x_n]^T$, $(m < n)$

1.3.1 Method of constrained variation

To understand the salient features of the method we consider the simple problem

Minimize $f(x_1, x_2)$

subject to $g(x_1, x_2) = 0$

Let us assume that $g(x_1, x_2) = 0$ can be solved to obtain x_2 as $x_2 = h(x_1)$. Then the problem reduces to the unconstrained minimization problem

Minimize $f(x_1, h(x_1))$

The necessary condition gives

$$\frac{df}{dx_1} = 0$$

or, $$\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial h}\,\frac{dh}{dx_1}=0$$

or, $$\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial x_2}\,\frac{dx_2}{dx_1}=0$$

or, $$\frac{\partial f}{\partial x_1}dx_1+\frac{\partial f}{\partial x_2}dx_2=0 \quad \text{.............(1)}$$

Let (x_1^*, x_2^*) be the minimum point. Then (x_1^*, x_2^*) must satisfy the given constraint.

$$\therefore \quad (x_1^*, x_2^*) = 0 \quad \text{............... (2)}$$

For admissible variations dx_1, dx_2 we have $g\,(x_1^* + dx_1, x_2^* + dx_2) = 0$

Using Taylor's theorem we get

$$g(x_1^*, x_2^*)+\left[\frac{\partial g}{\partial x_1}\right]_{(x_1^*, x_2^*)}dx_1+\left[\frac{\partial g}{\partial x_2}\right]_{(x_1^*, x_2^*)}dx_2 = 0$$

or, $$\left[\frac{\partial g}{\partial x_1}\right]_{(x_1^*, x_2^*)}dx_1+\left[\frac{\partial g}{\partial x_2}\right]_{(x_1^*, x_2^*)}dx_2=0 \quad \text{[by (2)]}$$

Assuming $\left[\frac{\partial g}{\partial x_2}\right]_{(x_1^*, x_2^*)} \neq 0$ we get,

$$dx_2 = -\frac{\left[\frac{\partial g}{\partial x_2}\right]_{(x_1^*, x_2^*)}}{\left[\frac{\partial g}{\partial x_2}\right]_{(x_1^*, x_2^*)}}dx_1 \quad \text{...........(3)}$$

Thus the admissible variation dx_2 depends on dx1and dx_1 can be chosen arlitravity.

Using (3) in (1) we have for admissible unviations

$$\left[\frac{\partial f}{\partial x_1}-\frac{\frac{\partial g}{\partial x_1}}{\frac{\partial g}{\partial x_2}}\cdot\frac{\partial f}{\partial x_2}\right]_{(x_1^*, x_2^*)}dx_1 = 0$$

Since dx_1 is albitrary we have

$$\left[\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2}-\frac{\partial g}{\partial x_1}\cdot\frac{\partial f}{\partial x_2}\right]_{(x_1^*,x_2^*)} = 0$$

This is the necessary condition for (x_1^*, x_2^*) to be an extreme point.

Result : The solution of the problem

Minimize $f(x_1, x_2)$

subject to $g(x_1, x_2) = 0$

is obtained by sloving

$$\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2}-\frac{\partial g}{\partial x_1}\frac{\partial f}{\partial x_2}=0$$

and $g\ (x_1, x_2) = 0$

The above result can be generalized for general problem in the following theorem.

Theorem 1.3.1. Necessary conditions for $(x_1^*, x_2^*, \ldots\ldots x_n^*)$ to be an extreme point of the function $f(x_1, x_2, \ldots\ldots, x_n)$ to exist under the m equality constraints $g_j\ (x_1, x_2, \ldots\ldots x_n) = 0$, $j = 1, 2 \ldots\ldots, m\ (m < n)$ are the following $(n - m)$ equations ar satisfied at $(x_1^*, x_2^*, \ldots\ldots x_n^*)$.

$$J\left(\frac{f, g_1, g_2, \ldots\ldots, g_m}{x_k, x_1, x_2, \ldots\ldots, x_m}\right) = \begin{vmatrix} \frac{\partial f}{\partial x_k} & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_m} \\ \frac{\partial g_1}{\partial x_k} & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_m} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial g_m}{\partial x_k} & \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_m} \end{vmatrix} = 0$$

$k = m + 1, m + 2, \ldots\ldots n$

$$\text{Where } J\left(\frac{g_1, g_2, \ldots\ldots, g_m}{x_1, x_2, \ldots\ldots, x_m}\right) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_m} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_m} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_m} \end{vmatrix} \neq 0$$

Note : In the above theorem $x_{m+1}, x_{m+2}, \ldots\ldots, x_n$ are independent variables. Also we note that the dependent variable, $x_1, x_2, \ldots\ldots x_m$ must satisfy J $\left(\frac{g_1, g_2, \ldots\ldots g_m}{x_1, x_2, \ldots, x_m}\right) \neq 0$

Example 1.3.1 Using method of constrained variation

Minimize $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + x_3^2$

subject to $2x_1 + 4x_2 + 3x_3 + 9$

$4x_1 + 8x_2 + 5x_3 + 17$

Solution.

We are b minimize

$$f = x_1^2 + 2x_2^2 + x_3^2$$

subject to $g_1 = 2x_1 + 4x_2 + 3x_3 - 9 = 0$ ....(1)

$g_2 = 4x_1 + 8x_2 + 5x_3 - 17 = 0$ ....(2)

We are first to select independent and dependent variable.

Let us consider

$$J\left(\frac{g_1, g_2}{x_1, x_2}\right) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 4 & 8 \end{vmatrix} = 0$$

Thus x_3 cannot be chosen as independent variables.

Let us now conisder

$$J\left(\frac{g_1, g_2}{x_1, x_2}\right) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 10 - 12 = -2 \neq 0$$

Thus x_2 cannot be chosen as independent variables.

The necessary condition is

$$J\left(\frac{f, g_1, g_2}{x_2, x_1, x_3}\right) = 0$$

or, $\begin{vmatrix} \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_3} \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_3} \end{vmatrix} = 0$

or, $\begin{vmatrix} 4x_2 & 2x_1 & 2x_3 \\ 4 & 2 & 3 \\ 8 & 4 & 5 \end{vmatrix} = 0$

or, $4x_2\ (10 - 12) + 2x_1\ (24 - 20) + 2x_3\ (16 - 16) = 0$

or, $-8x_2 + 8x_1 + 0 = 0$

or, $x_2 = x_1$..........(3)

Using (3) in (1) & (2) we get respectively

$6x_1 + 3x_3 - 9 = 0$

and $12x_1 + 5x_3 - 17 = 0$

$\therefore\quad x_1 = \frac{-15+45}{30-36} = 1$

$x_3 = \frac{-108+102}{30-36} = 1$

From (3) we have $x_2 = 1$

Hence the required solution is $x_1 = 1,\ x_2 = 1,\ x_3 = 1$.

1.3.2 Method of Lagrange multipliers

In the Lagrange miltiplier method are additional variable is introduced to the problem for each constraint. If the original problem has n variables and m equality constraints then we are to add m additional variables to the problem so that the final number of unknowns becomes n + m.

We now state the famous theorems of Lagrange.

Theorem 1.3.2 A necessary condition for a function $f(x_1, x_2, ..., x_n)$ subject to the constraints $g_j\ (x_1, x_2, ..., x_n) = 0$, j = 1, 2, ..., m to have a relative minimum at a point $(x_1^*, x_2^*, ... x_n^*)$ is that the first partial derivatives of the Lagrange function

$L = (x_1, x_2, \ldots\ldots, x_n, \lambda_1, \lambda_2, \ldots\ldots \lambda_n) = f + \sum_{j=1} \lambda_j g_j$ with respect to each of its arguments must be zero.

The sufficient condition for a function subject to equality constranits is given in the following theorem.

Theorem 1.3.3 A sufficient condition for a function $f(x_1, x_2, \ldots, x_n)$ subject to the constraints $gj(x, x_2, \ldots\ldots, x_n) = 0$, $j = 1, 2, \ldots\ldots, m$ to have a relative minimum (maximum) at a point $(x_1^*, x_2^*, \ldots\ldots x_n^*)$ is that the quadratic Q, defined by

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 L}{\partial x_i \partial x_j} dx_i dx_j \ \ldots\ldots\ldots\ldots (\ ) \text{ evaluated at } (x_1^*, x_2^*, \ldots\ldots x_n^*) \text{ must be positive}$$

(negative) definite for all choice of admissible variations dx_i.

Theorem (Hanock) 1.3.4 A necessary condition for the quadratic form $Q = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 L}{\partial x_i \partial x_j} dx_i dx_j$; evaluated at $(x_1^*, x_2^*, \ldots\ldots x_n^*)$ to be positive (negative) definite for all admissible variations dx_i is that each root of the polynomial defined by the following determinantal equation, be positive (negative) :

$$\begin{vmatrix}
(L_{11} - Z) & L_{12} & \cdots & \cdots & L_{1n} & g_{11} & g_{21} & \cdots & \cdots & g_{m1} \\
L_{21} & (L_{22} - Z) & \cdots & \cdots & L_{2n} & g_{12} & g_{22} & \cdots & \cdots & g_{m2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
L_{n1} & L_{n2} & \cdots & \cdots & (L_{nn} - Z) & g_{1m} & g_{2m} & \cdots & \cdots & g_{mm} \\
g_{11} & g_{12} & \cdots & \cdots & g_{1m} & 0 & 0 & \cdots & \cdots & 0 \\
g_{21} & g_{22} & \cdots & \cdots & g_{2m} & 0 & 0 & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
g_{m1} & g_{m2} & \cdots & \cdots & g_{mm} & 0 & 0 & \cdots & \cdots & 0
\end{vmatrix}$$

Where $L_{ij} = \left[\frac{\partial^2 L}{\partial x_i \partial x_j}\right]_{X^*}$

and $g_{ij} = \left[\frac{\partial g_i}{\partial x_j}\right]_{X^*}$, $X^* = (x_1^*, x_2^*, \ldots\ldots x_n^*)$

Result : If some of the roots of the above determinantal equation are positive and some are negative then the point x^* is not an extreme point.

Example 1.3.2 : Using Lagrange multiplier method minimize the function

$f(x_1, x_2, x_3) = 9 - 8x_1 - 6x_2 - 4x_3 + 2x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3$ subjectto the constrain $x_1 + x_2 + 2x_3 = 3$

Solution. Hope $f = 9 - 8x_1 - 6x_2 + 4x_3 + 2x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3$

$$g = x_1 - x_2 + 2x_3 - 3 = 0$$

The Lagrange functionis given by

$L(x_1, x_2, x_3, \lambda) = f + \lambda g$

$= (9 - 8x_1 - 6x_2 + 4x_3 + 2x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3) + \lambda\ (x_1, x_2, 2x_3 - 3)$

The necessary condition are

$$\frac{\partial L}{\partial x_1} = 0 \qquad \text{or.} \quad -8 + 4x_1 + 2x_2 + 2x_3 + \lambda = 0$$

$$\frac{\partial L}{\partial x_2} = 0 \qquad \text{or,} \quad -6 + 4x_2 + 2x_1 + \lambda = 0$$

$$\frac{\partial L}{\partial x_3} = 0 \qquad \text{or,} \quad -4 + 2x_3 + 2x_1 + 2\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 0 \qquad \text{or,} \quad x_1 + x_2 + 2x_3 - 3 = 0$$

Solving these four equations we have

$$x_1^* = 4/3,\ x_2^* = 7/9,\ x_3^* = 4/9 \text{ and } \lambda^* = 2/9$$

We now use sufficient condition to identify this extreme point.

We evaluate L_{ij} and g_{ij} at the point $(4/3, 7/9, 4/9) = X^*$

$$L_{11} = \left[\frac{\partial^2 L}{\partial x_1^2}\right]_{X^*} = 4$$

$$L_{12} = L_{21} = \left[\frac{\partial^2 L}{\partial x_1 x_2}\right]_{X^*} = 2$$

$$L_{13} = L_{31} = \left[\frac{\partial^2 L}{\partial x_1 x_3}\right]_{X^*} = 2$$

$$L_{22} = \left[\frac{\partial^2 L}{\partial x_2^2}\right]_{X^*} = 4$$

$$L_{23} = L_{32} = \left[\frac{\partial^2 L}{\partial x_2 x_3}\right]_{X^*} = 0$$

$$L_{33} = \left[\frac{\partial^2 L}{\partial x_3^2}\right]_{X^*} = 2$$

$$g_{11} = \left[\frac{\partial g}{\partial x_1}\right]_{X^*} = 1$$

$$g_{12} = \left[\frac{\partial g}{\partial x_2}\right]_{X^*} = 1$$

$$g_{13} = \left[\frac{\partial g}{\partial x_3}\right]_{X^*} = 2$$

We now consider the determinautor equation

$$\begin{vmatrix} L_{11}-z & L_{12} & L_{13} & g_{11} \\ L_{21} & L_{22}-z & L_{23} & g_{12} \\ L_{31} & L_{32} & L_{33}-z & g_{13} \\ g_{11} & g_{12} & g_{13} & 0 \end{vmatrix} = 0$$

or, $$\begin{vmatrix} 4-z & 2 & 2 & 1 \\ 2 & 4-z & 0 & 1 \\ 2 & 0 & 2-z & 2 \\ 1 & 1 & 2 & 0 \end{vmatrix} = 0$$

or, $$-1\begin{vmatrix} 2 & 2 & 1 \\ 4-z & 0 & 1 \\ 0 & 2-z & 2 \end{vmatrix} + 1\begin{vmatrix} 4-z & 2 & 2 \\ 2 & 0 & 2-z \\ 1 & 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 4-z & 2 & 2 \\ 2 & 4-z & 0 \\ 1 & 1 & 2 \end{vmatrix} = 0$$

or, $z^2 - 6z + 9 = 0$

or, $z = 3, 3$

Since the roots are all positive, (4/3, 7/9, 4/9) is a relative minimum of the function.

1.4 Multivariable optimization with inequality constraints

The general multivariable optimization problem with inequality constraints is

Minimize $f = f(X)$

subject to $g_i(x) \le b_j \quad j = 1, 2, \ldots, m$

where $X = [x_1, x_2, \ldots\ldots x_n]^T$.

This section is concerned with developing the necessary and sufficient conditions for identifying the stationery points of the above problem. These conditions are called Kuhn-Tucker conditions and the development is mainly based on Lagrangian method.

Theorem 1.4.1 (Kuhn-Tucker Necessary Conditions)

Given the problem to minimize

$$f = f(x) = f(x_1, x_2, \ldots, x_n)$$

subject to $gj(X) = g_j(x_1, x_2, \ldots, x_n) \le b_j \quad i = 1, 2, \ldots, m$

the necessary conditions for X_0 to be a local minimum are that

(i) $\dfrac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \dfrac{\partial g_j}{\partial x_i} = 0, \quad i = 1, 2, \ldots n$

(ii) $\lambda_j [g_j(X) - b_j] = 0, \quad j = 1, 2, \ldots m$

(iii) $g_j(X) \le b_j, \quad j = 1, 2, \ldots m$

(iv) $\lambda_j \ge 0, \quad j = 1, 2, \ldots, m$

are satisfied at X_0.

Introducing slack variables the inequality constraints bcomes

$$g_j(X) + s_j^2 = b_j, \quad j = 1, 2, \ldots, m$$

or, $g_j(X) + s_j^2 - b_j = 0, \quad j = 1, 2, \ldots m. \quad \ldots\ldots(1)$

In order to obtain all stationary points, we form the Lograngian function L given by

$$L\ (X,\ l,\ S) = f(X) + \sum_{j=1}^{m} \lambda_j \{g^j\ (X) + s_j^2 - b^j\}$$

Then the stationary points are obtained by polving the equations

$$\frac{\partial L}{\partial x_i} = 0,\ i = 1, 2, \ldots\ldots, n$$

$$\frac{\partial L}{\partial \lambda_j} = 0,\ j = 1, 2, \ldots\ldots, m$$

and $\frac{\partial L}{\partial s_j} = 0,\ j = 1, 2, \ldots\ldots, m$

i.e., $\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0,\quad i = 1, 2, \ldots\ldots, n$ ....(2)

$g_j + s_j^2 - b_j = 0$ ....(3)

$2\lambda_j s_j = 0,$ ....(4)

Multiplying (4) by $\frac{1}{2}s$; we get,

$$\lambda_j s_j^2 = 0$$

Using (1) this gives

$$\lambda_j \{b_j - g_j(X)\} = 0$$

or, $\lambda_j \{g_j(X) - b_j\} = 0,\ \ j = 1, 2, \ldots\ldots\ m$ ....(5)

From (5) we have when $\lambda_j \neq 0$ then $g_j\ (X) - b_j = 0$ or, $g_j\ (X) = b_j$

or, $\frac{\partial g_j}{\partial b_j} = 1$

Thus $\frac{\partial g_k}{\partial b_j} = s_{jk}$ where $s_{jk} = \begin{cases} 1\ for\ j = k \\ 0\ for\ j \neq k \end{cases}$

Using chain rule of differential calculus we have

$$s_{jk} = \frac{\partial g_k}{\partial b_j} = \sum_{i=1}^{n} \frac{\partial g_k}{\partial x_i} \frac{\partial x_i}{\partial b_j}$$

Multiplying both sides by λk and summing over all values of k we get

$$\sum_{k=1}^{m} \lambda_k s_{jk} = \sum_{k=1}^{m} \lambda_k \left(\sum_{i=1}^{n} \frac{\partial g_k}{\partial x_i} \frac{\partial x_i}{\partial b_j} \right)$$

or, $$\lambda_j = \sum_{k=1}^{m} \lambda_* \left(\sum_{i=1}^{n} \frac{\partial g_k}{\partial x_i} \frac{\partial x_i^*}{\partial b_j} \right) \qquad \text{............(6)}$$

Again $$\frac{\partial f}{\partial b_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial b_j} \qquad \text{............(7)}$$

Adding (6) and (7) we get

$$\frac{\partial f}{\partial b_j} + \lambda_j = \sum_{i=1}^{n} \left[\frac{\partial f}{\partial x_i} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial x_i} \right] \frac{\partial x_i}{\partial b_j}$$

$= 0$ [using (2)]

or, $$\frac{\partial f}{\partial b_j} = -\lambda_j \qquad \text{...............(8)}$$

Thus when $\lambda_j \neq 0$ then we have $\lambda_j = -\dfrac{\partial f}{\partial b_j}$ ....(9)

We now show that $\lambda_j > 0$. If possible let $\lambda_j < 0$. Then from (9) we have $\dfrac{\partial f}{\partial b_j} > 0$

This implies that as b_j is increased, the objective function increases. Now as b_j optimal value of the objective function clearly cannot increase. This contradicts our assumption $\lambda_j > 0$. Thus at an optimal solution we have $\lambda_j > 0$ when $\lambda_j \neq 0$. Hence at the optimal solution we hour $\lambda j \geq 0$.

Note : For the problem

Maximize $f = f(x_1, x_2, \ldots\ldots, x_n)$

subject to $g_j(x_1, x_2, \ldots\ldots, x_n)\ b_j$, $i = 1, 2, \ldots\ldots m$

the Kuhn-Tucker necessary conditions for $(x_1^*, x_2^*, \ldots\ldots x_n^*)$ to be a local maximum are that

(i) $\frac{\partial f}{dx_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_i}{\partial x_i} = 0, \; i = 1, 2, \ldots\ldots, n$

(ii) $\lambda_j \; [g_j - b_j] = 0, \; j = 1, 2, \ldots\ldots, m$

(iii) $g_j \leq b_j, \; j = 1, 2, \ldots\ldots, m$

(iv) $\lambda_j \; 0, \; j = 1, 2, \ldots\ldots, m$

are satified at $(x_1^*, x_2^*, \ldots\ldots x_n^*)$

Sufficency of the Kuhn-Tucker conditions

The Kuhn-Tucker necessary conditions are also sufficient if the objective function and the solution space satisfy certain conditions regarding convexity anc concairty. For maximization problem the objective function should be concave and solution space should be convex set.

For minimization problem the objective function should be convex and the solution space should be convex set.

Example 1.4.1. Solve using Kuhn-Tucker conditions

Maximize $z = 5 + 8x_1 + 12x_2 - 4x_1^2 - 4x_2^2 - 4x_3^2$

subject to $x_1 + x_2 \leq 1$

$2x_1 + 3x_2 \leq 6$

Here the constraints are

$g_1 = x_1 + x_2 \leq 1$

and $g_2 = 2x_1 + 3x_2 \leq 6$

The Kuhn-Tucker necessary conditions are

$$\frac{\partial Z}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} + \lambda_2 \frac{\partial g_2}{\partial x_i} = 0, \; i = 1, 2, 3$$

$\lambda_j \; [g_j - b_j] = 0, \; j = 1, 2$

$\lambda_j \leq 0, \; j = 1, 2$

i,e., $8 - 8x_1 + \lambda_1 + 2\lambda_2 = 0$ ....(1)

$$12 - 8x_2 + \lambda_1 + 3\lambda_2 = 0 \quad \text{........(2)}$$

$$- 8x_1 = 0 \quad \text{........(3)}$$

$$\lambda_1 + (x_1 + x_2 - 1) = 0 \quad \text{........(4)}$$

$$\lambda_2 + (2x_1 + 3x_2 - 6) = 0 \quad \text{........(5)}$$

$$x_1 + x_2 - 1 = 0 \quad \text{........(6)}$$

$$2x_1 + 3x_2 - 6 \leq 0 \quad \text{........(7)}$$

$$\lambda_1 \leq 0 \quad \text{........(8)}$$

$$\lambda_2 \leq 0 \quad \text{........(9)}$$

Four cases may arise.

case 1. $\lambda_1 = 0, \lambda_2 = 0$

case 2. $\lambda_1 = 0, \lambda_2 \neq 0$

case 3. $\lambda_1 \neq 0, \lambda_2 = 0$

case 4. $\lambda_1 \neq 0, \lambda_2 \neq 0$

Case 1. Here $\lambda_1 = 0, \lambda_2 = 0$

From (1) we get $x_1 = 1$

From (2) we get $x_2 = 3/2$

This solution does not satisfy (6). So this solution is discarded

Case 1. Here $\lambda_1 = 0, \lambda_2 \neq 0$

From (5) we get $2x_1 + 3x_1 - 6 = 0$ 10

(1) becomes $8 - 8x_1 + 2\lambda_2 = 0$ or, $x_1 = (\lambda_2 + 4)/4$ ....(11)

(2) becomes $12 - 8x_2 + 3\lambda_2 = 0$ or, $x_2 = (3\lambda_2 + 12)/8$ ....(11)

Using (11) and (12) we get from (10)

$$(\lambda_2 + 4)/2 + (g\lambda_2 + 36)/S - 6 = 0$$

or, $4\lambda_2 + 16 + g\lambda_2 + 36 - 48 = 0$

or, $13\lambda_2 = - 4$

or, $l2 = -4/13 < 0$

From (11) we have $x_1 = -\frac{1}{13} + 1 = \frac{12}{13}$

From (11) we have $x_1 = -\frac{24}{104} + \frac{12}{18} = \frac{18}{13}$

This solution violets (6) and so is discorded.

Case. 3 Here $\lambda_1 \neq 0$ and $\lambda_2 = 0$

From (4) we have $x_1 + x_2 - 1 = 0$ ....(13)

(1) becomes $8 - 8x_1 + \lambda_1 = 0$ or, $x_1 = (\lambda_1 + 8)/8$ ....(14)

(2) becomes $12 - 8x_2 + \lambda_1 = 0$ or, $x_2 = (\lambda_1 + 12)/8$ ....(14)

Using (14), (15) in (13) we have

or, $\lambda_1 = -6$

From (14) and (15) we get $x_1 = 1/4$, $x_2 = 3/4$

From (3) we get $x_3 = 0$

$\therefore \quad x_1 = 1/4, x_2 = 3/4, x_3 = 0$

This solution satifies (6) and (7).

Hence this is the optimum solution.

1.5 Summary

This unit is devoted with the classical theory of optimization for locating the points of maxima and minima of constrained and unconstrained nonlinear problems. This theory deals with the use of diffrential calculus. The topics introduced includs the development of the necessary and sufficient conditions for locating the extreme points for unconstrained problems, the treatment of the constrained problem with equality constraints using Lagrangian methods, and the development of the Kuhn-Tucker conditions for the general problem with inequality constraints. Though the classical optimization techniques are not suitable for obtaining real life problems, the underlying theory gives the basis for devising most of the non-linear programming algorithms.

1.6 Assesment Questions

1. Determine the extreme points of the function

$$f = 8x_1^3 + 27x_2^3 + 16x_1^2 + 18x_2^2 + 6$$

2. Determine the extreme points of the function

$$Z = 121 + 27x_1^3 + 64x_2^3 + 36x_1^2 + 32x_2^2$$

3. Find the extreme points of the function

$$f = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2 + 20$$

4. The total profits (z) of a firm depend upon the level of of output (Q) and the advertising expenditure (A). Find the profit maximizing values of Q (in thousand units) and A (Rs in thousand) given the following relationship.

$$Z = 800 - 3Q^2 - 4Q + 2QA - 5A^2 + 48A$$

5. Using method of constrained variation and method of Lagrange multiplier

(i) Minimize $f(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$

Subject to $x_1 = x_2$

$x_1 + x_2 + x_3 = 1$

(ii) Minimize $f = 19 - 16x_1 + 6x_2 - 4x_3 + 8x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2 + 4x_1x_3$

subject to $2x_1 - x_2 + 2x_3 = 3$

(iii) Maximize $f = 8x_1 x_2 x_3$

subject to $x_1^2 + x_2^2 + x_3^2 = 1$

(iv) Minimize $f = 4x_1^2 + 2x_2^2 + 9x_3^2$

subject to $4x_1 - 4x_2 + 9x_3 = 9$

$8x_1 - 8x_2 + 15x_3 = 17$

6. Using Kunh-Tucker condition determine the variable values to

Maximize $z = x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$

subject to $x^1 + x_2 \leq 2$

$2x_1 + 3x_2 \leq 12$

7. Use Kuhn-Tucker conditions of solve the following non-linear programming problems

(i) Maximize $Z = x_1^2 + 6x_1 + 5x_2$

subject to $x_1 + 2x_2 \leq 10$

$x_1 + 3x_2 \leq 9$

(ii) Maximize $Z = 2x_1 - x_1^2 + x_2$

subject to $2x_1 + 3x_2 \leq 6$

$2x_1 + x_2 \leq 4$

(iii) Maximize $Z = 2x_1^2 + 12x_1x_2 - 7x_2^2$

subject to $2x_1 + 5x_2 \leq 98$

$x_1 + x_2 \geq 0$

(iv) Maximize $Z = 8x_1 + 10x_2 - x_1^2 - x_2^2$

subject to $3x_1 + 2x_2 \leq 6$

$x_1, x_2 \geq 0$

Unit 2 ☐ Revised Simplex Method

Structure

2.1 Introduction :

The revised simplex method proceeds through the same steps as simplex method but keeps all important data in a smaller array. The 'revised' aspect concerns the procedure of changing the simplex tables only. The revised simplex method is thus an efficient computational procedure for solving a linear programming problem with less time and labour. For large size problem this method is found to be ..... useful as it reduces the cost of obtaining the solution.

2.2 Revised Simplex Method :

When a linear programming problem is solved simplex method, successive iterations are obtained by using suitable row operations so that the objective function reduces its value in each step if it is a problem of maximization. Also the net evaluations should remain always non-negative in every step. This method requires storing the entire table in the memory of the computer. For large size problem it may not be feasible. So, it requires to device a new method by modifying simplex method to handle LPP will large number of decision variables and constraints.

In fact, it is found tat it is not necessary to compute the entire simplex table during each iteration. The only informations needed to pass from one table to the next one are seen to be

(i) Net evaluations $z_j - e_j$ to determine the non-basic variable that enters the basis.

(ii) The key column.

(iii) The current basic variables and their values to determine the minimum positive ratio, and thereby to determine the basic variable that will leave the basis.

It is sown that all the above informations can be directly obtained from the original equations of the given LPP by making use of the inverse of the current basis matrix.

If B be the current basis then we have

$$x_B = B^{-1} b, y_j = B^{-1} a_j \text{ for all } j = 1, 2, ..., n$$

$$z_j - c_j = C_B B^{-1} a_j - c_j \text{ for all } j = 1, 2, ..., n$$

and $z = c_B x_B$.

We note that all these necessary informations can be calculated if the current value of B^{-1} is known. Much computational work is needed for transformation of all y_j, $j = 1, 2, ..., n$.

But all y; are not needed to go to next table. As noted above we need only to know the key column i.e. y_k. This will actively save our much labour. At each iteration x_B, z $c_B B^{-1}$ and B^{-1} are transformed and not all the y_j are transformed, only the key column y_k is transformed in the revised simplex method. The criteria for selecting the entering and departing vectors in the revised simplex method precisely the same as that was in the simplex method. The labour saving point in this method lies in the fact of computing the inverse of the next basis directly from that of the current basis without actually having to invert the next basis.

2.3 Standard Form for Revised Simplex Method :

Let the linear programming problem be

$$\begin{aligned} &\text{Maximize} && z = cx \\ &\text{subject to} && Ax = b \\ &&& x \geq 0 \end{aligned} \qquad \text{....... (1)}$$

where $c, x^T \in R^n$, $b^T \in R^m$ and A is an $m \times n$ real matrix. In the revised simplex method we consider the objective function equation $z = cx$ also one constraint. Thus the new system becomes a $(m + 1)$ simultaneous lines equations in $n + 1$ variables z, $x_1, x_2, \ldots, x_n$. The problem thus becomes to get the solution of this system such that z is as large as possible. The simultaneous linear system thus becomes

$$\begin{aligned} &Ax + oz = b \\ &-cx + z = 0 \\ &\quad x \geq 0, z \text{ is unrestricted.} \end{aligned} \qquad \text{....... (2)}$$

Hence the LPP (1) becomes equivalent to the problem of finding the solution of the system (2) such that z is as large as possible.

In matrix notation (2) becomes

$$\begin{bmatrix} A & 0 \\ -c & 1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}, x \geq 0 \qquad \text{....... (3)}$$

Let B be the initial basis submatrix of A and $x_B = B^{-1}b$ be the initial basic feasible solution to the original LPP (1).

Since the values of the non-basic variables are always zero (2) becomes

$$\begin{aligned} &Bx_B + 0\,z = b \\ &-C_B\, x_B + z = 0 \end{aligned} \qquad \text{....... (4)}$$

$$\text{or, } \begin{bmatrix} B & 0 \\ -C_B & 1 \end{bmatrix} \begin{bmatrix} x_B \\ z \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$$\text{or, } \hat{B}\,\hat{x}_B = \hat{b} \qquad \text{....... (5)}$$

where, $\hat{B} = \begin{bmatrix} B & 0 \\ -C_B & 1 \end{bmatrix}, \hat{x}_B = \begin{bmatrix} x_B \\ z \end{bmatrix}$ and $\hat{b} = \begin{bmatrix} b \\ 0 \end{bmatrix}$ .... (6)

From (4) we have

$$\hat{x}_B = \hat{B}^{-1}\hat{b} \qquad \text{....... (7)}$$

This is the initial basic feasible solution to the reformulated problem (2).

Computation of Inverse of $\hat{B}$ by partitioning we have $B = \begin{bmatrix} B & 0 \\ -C_B & 1 \end{bmatrix}$.

Let $\hat{B}^{-1} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ .... (8)

Since $\hat{B}\hat{B}^{-1} = I$, we have

$$\begin{bmatrix} B & 0 \\ -C_B & 1 \end{bmatrix}\begin{bmatrix} P & Q \\ R & S \end{bmatrix} = I_{m+1}$$

or, $\begin{bmatrix} BP+OR & BQ+OS \\ -C_BP+R & -C_BQS \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore$ $BP = I_m$

$BQ = O$

$-C_B P + R = 0$

$-C_B Q + S = 1$

Since B^{-1} exists, we get from above

$P = B^{-1} I_m = B^{-1}$

$Q = B^{-1} O = O$

$R = C_B B^{-1}$

$S = 1 + C_BO = 1$

Thus from (8) we get

$$B^{-1} = \begin{bmatrix} B^{-1} & 0 \\ C_BB^{-1} & 1 \end{bmatrix} \qquad \text{....... (9)}$$

We note that all the components of $\hat{B}^{-1}$ are known.

Determination of net evaluations, key column and BFS :

We define $A = \begin{bmatrix} A \\ -C \end{bmatrix}$

and $\hat{y} = \hat{B}^{-1}\hat{A}$

Then $\hat{y} = \begin{bmatrix} B & 0 \\ C_B B^{-1} & 1 \end{bmatrix} \begin{bmatrix} A \\ -C \end{bmatrix}$

$$= \begin{bmatrix} B^{-1}A & -OC \\ C_B B^{-1}A & -C \end{bmatrix}$$

$$= \begin{bmatrix} B^{-1}A \\ C_B(B^{-1}A) - C \end{bmatrix} \quad \text{....... (10)}$$

we have

$A = By$

$\therefore\ y = B^{-1}A$

$\therefore$ From (10) we have $\hat{y} = \begin{bmatrix} y \\ C_B\, y - C \end{bmatrix}$

or, $[\hat{y}_1\ \hat{y}_2 \ldots\ldots \hat{y}_n] = \begin{bmatrix} y_1 & y_2 & \ldots\ldots & y_n \\ z_1 - c_1 & z_2 - c_2 & \ldots\ldots & z_n - c_n \end{bmatrix}$

Thus for $j = 1, 2, \ldots\ldots, n$ we have

$$\hat{y}_j = \begin{bmatrix} y_j \\ z_j - c_j \end{bmatrix} \text{ and } y_j = B^{-1} a_j$$

Hence the net evaluation are the components of $C_B B^{-1}A - C$

i.e. $C_B B^{-1}A - C = [z_1 - c_1\ \ z_2 - c_2 \ldots\ldots z_n - C_n]$

Most negative $z_j - c_j$ will determine the key column. Let $z_k - c_k$ be the most negative $z_j - c_j$. Then the key column is

$$\hat{y}_k = \begin{bmatrix} y_k \\ z_k - c_k \end{bmatrix} = \begin{bmatrix} B^{-1} a_k \\ z_k - c_k \end{bmatrix} \quad \text{....... (11)}$$

From (7) and (6) we have

$$\hat{x}_B = \begin{bmatrix} x_B \\ z \end{bmatrix} = \hat{B}^{-1} \hat{b} = \begin{bmatrix} B^{-1} & 0 \\ C_B B^{-1} & 1 \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} B^{-1} b \\ C_B B^{-1} b \end{bmatrix} = \begin{bmatrix} B^{-1} b \\ C_B x_B \end{bmatrix}$$

we note the important fact that all necessary informations can be obtained from the products $\hat{B}^{-1}\hat{A}$ and $\hat{B}^{-1}\hat{b}$.

Also we note that $\hat{A}$ and $\hat{b}$ remains same in all steps, only $\hat{B}^{-1}$ changes in each step of simplex table depending on the current basis B.

The above discussion enables us now to state the algorithm of revised simplex method.

2.4 ALogarithm of Revised Simplex Method :

It stepwise procedure of revised simplex method are as follows.

Step 1. Introduce necessary slack and surplus variables. Convert the problem into a problem of maximization if it is in minimization form. Restate the LPP in the standard form of revised simplex method *i.e.* in the form $\begin{bmatrix} A & 0 \\ -c & 1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}, x \geq 0, z$ is unrestricted.

Step 2. Begin with the initial basis $B = I_m$ and form the auxiliary matrix $\hat{B} = \begin{bmatrix} B & 0 \\ -C_B & 1 \end{bmatrix}$ and write down

$$B^{-1} = \begin{bmatrix} B^{-1} & 0 \\ C_B B^{-1} & 1 \end{bmatrix}. \text{ Form } \hat{A} = \begin{bmatrix} A \\ -c \end{bmatrix} \text{ and } \hat{b} = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

Also form $\hat{x}_B = \begin{bmatrix} x_B \\ z \end{bmatrix} = \hat{B}^{-1} \hat{b}$.

Step 3. Compute the net evaluations $z_1 - c_1, z_2 - c_2, \ldots .. z_n - c_n$ as the components of the product

$$[C_B B^{-1} \quad 1]\begin{bmatrix} A \\ -c \end{bmatrix}$$

If all $z_j - c_j$ are non-negative, the current basic solution $\hat{x}_B = \begin{bmatrix} x_B \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b}$ gives the optimal BFS and maximum value of the objective function.

If at least one $z_j - c_j$ is negative, determine the most negative of them. If $z_k - c_k$ is the most negative $z_j - c_j$ then find $\hat{y}_k = \begin{bmatrix} y_k \\ z_k - c_k \end{bmatrix} = \hat{B}^{-1}\hat{a}_k$. Go to step 4. If there is a tie for the most negative $z_j - c_j$, resolve the tie by any standard method.

Take x_k as the new basic variable. Go to step 4.

Step 4. If all $y_{ik} \leq 0$ there exists an unbounded solution to the given problem.

If at least one $y_{ik} > 0$, consider the current x_B and compute the replacement ratios.

$$\left\{\frac{x_{Bi}}{y_{ik}} : y_{ik} > 0\right\}$$

If $\frac{x_{Br}}{y_{rk}}$ is the minimum of all these ratios then the basic variable x_{Br} becomes non-basic variable in the next table. *ie.* x_{Br} is replaced by x_k. Go to step 5.

Step 5. Write down the results obtained in steps 2, 3 and 4 in a able. This table is known as revised simplex table. This table is of the form

$\hat{y}_B$	$\hat{x}_B$	$\hat{B}^{-1}$	$\hat{y}_k$	$\frac{x_{Bi}}{y_{ik}}; y_{ik>0}$

Step 6. Convert the key element y_{rk} of $\hat{y}_k$ into unity and all other elements into zero by suitable row operations. Same operations are to be applied in the current $\hat{B}^{-1}$. These operation will change $\hat{B}^{-1}$ to new $\hat{B}^{-1}$ for the next table.

Step 7. Consider new $\hat{B}^{-1}$ obtained in step 6 as $\hat{B}^{-1}$ and go to step 3. Repeat the procedure until an optimum basic feasible solution is obtained or there is an indication of an unbounded solution.

Advantages of revised simplex method :

The advantages of the revised simplex method over the regular simplex method are

(i) fewer calculations are required.

(ii) less storage is needed when computing the problem on a computer.

(iii) the round off errors can be controlled as table entries are not repeatedly recalculated.

2.5 Comparison of Simplex Method and Revised Simplex Method :

Let us consider the LPP

Maximize $z = cx$

subject to $Ax = b$, $x \geq 0$

where A is a matrix of order $m \times n$. If initially artificial variables are not needed for obtaining the initial basis matrix, then for solving this problem by the simplex method we have to transfer $(n + 1)$ columns at each iteration. (n columns for A and one column for x_B). Also, at each iteration one variable is introduced into the basis and one is removed from it. Thus, in total we compute for $(n - m + 1)$ columns. Further more, for each of these columns, we have to transform $(m + 1)$ elements. For moving from one iteration to another we also need to calculate minium ratio x_B/y_{ik}. Hence in all we have to perform multiplication $(m + 1)(n - m + 1)$ times and addition $m(n - m + 1)$ times.

In the revised simplex method, there are $(m + 1)$ rows and $(m + 2)$ columns. So, for moving from one iteration to another we have to make $(m + 1)^2$ multiplication operations to get an improved solution in addition to $m(n - m)$ operations for calculating $(z_j - c_j)$'s.

In the revised simplex method we need to make $(m + 1)(m + 2)$ entries in each table while in simplex method there are $(m + 1)(n + 1)$ entries in each table.

If the number of variables n is significantly larger than the number of constraints m, then the coputational efforts of the revised simplex method is smaller than that of the simplex method.

Revised simplex method reduces the cumulative round-off error while calculating $(z_j - c_j)$'s and updated column y_k due to the use of original data.

The inverse of the current basis matrix is obtained automatically.

2.6 Illustrative Examples :

Example 2.6.1. Use revised simplex method to solve the LPP.

Maximize $z = 2x_1 - 3x_2 + x_3$

subject to $3x_1 + 6x_2 + x_3 \leq 6$

$4x_1 + 2x_2 + x_3 \leq 4$

$x_1 - x_2 + x_3 \leq 3$

$x_1, x_2, x_3 \geq 0.$

Solution : Introducing slack variables $x_4 \geq 0, x_5 \geq 0, x_6 \geq 0$, the given LPP becomes in standard form as

Maximize $z = 2x_1 - 3x_2 + x_3 + 0x_4 + 0x_5 + 0x_6$

subject to $3x_1 + 6x_2 + x_3 + x_4 = x_6$

$4x_1 + 2x_2 + x_3 + x_5 = 4$

$x_1 - x_2 + x_3 + x_6 = 3$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

or, Maximize $z = cx$

subject to $Ax = b$, $x \geq 0$

where $A = \begin{bmatrix} 3 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{bmatrix}, c = 0\begin{bmatrix} 2 & -3 & 1 & 0 & 0 & 0 \end{bmatrix}$

$b = \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \, x_2 x_3 \, x_4 \, x_5 \, x_6 \end{bmatrix}^T$

$\therefore$ we have $\hat{A} = \begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} 3 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \\ -2 & 3 & -1 & 0 & 0 & 0 \end{bmatrix}, b = \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 3 \\ 0 \end{bmatrix}$

Initially

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, x_B = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix}, c_B = \begin{bmatrix} c_4 & c_5 & c_6 \end{bmatrix} = [0\ 0\ 0]$$

Now $C_B B^{-1} = [0\ 0\ 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [0\ 0\ 0]$

$$\therefore\ B^{-1} = \begin{bmatrix} B^{-1} & 0 \\ C_B B^{-1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore\ x_B = \begin{bmatrix} x_B \\ z \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ z \end{bmatrix} = B^{-1} b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 3 \\ 0 \end{bmatrix}$$

i.e. $\begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix}$ and $z = 0$

The net evaluation are the components of

$$[C_B B^{-1} \;\; 1]\begin{bmatrix} A \\ -c \end{bmatrix} = [0\;0\;0\;1]\begin{bmatrix} 3 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ -2 & 3 & -1 & 0 & 0 & 0 \\ -2 & 3 & -1 & 0 & 0 & 0 \end{bmatrix} = [-2\;3\;-1\;0\;0\;0]$$

$$= [z_1 - c_1 \;\; z_2 - c_2 \;\; z_3 - c_3 \;\; z_4 - c_4 \;\; z_5 - c_5 \;\; z_6 - c_6]$$

Since there are negative net evaluations, the solution obtained is not optimal. The most negative net evaluation is $z_1 - c_1 = -2$. Therefore x_1 will be the new basic variable.

Now we compute

$$y_1 = B^{-1} a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \\ -2 \end{bmatrix}$$

These results are shown in the following initial revised simplex table

Basic variables	Values	$\hat{B}^{-1}$				$\hat{y}^{-1}$	min ratio
x_4	6	1	0	0	0	3	2
x_5	4	0	1	0	0	4	1
x_6	3	0	0	1	0	1	3
z	0	0		0	1	– 2	

Here the minimum ratio is Min $\left\{\frac{x_{Bi}}{y_{ik}} : y_{ik} > 0\right\} = 1$ and the corresponding variable is x_5. Therefore, the outgoing basic variable is x_5. So x_5 is replaced by x_1 in the next table.

Using elementary row operations $\hat{y}_1 = \begin{bmatrix} 3 \\ 4 \\ 1 \\ -2 \end{bmatrix}$ is converted to $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and the same operations are done for $\hat{B}^{-1}$. This gives new $\hat{B}^{-1}$ as follows

$$\hat{B}^{-1} = \begin{bmatrix} 1 & -\frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{bmatrix}$$

The new BFS is given by

$$\hat{x} = \begin{bmatrix} x_4 \\ x_1 \\ x_6 \\ x \end{bmatrix} = \hat{B}^{-1}.\hat{b} = \begin{bmatrix} 1 & -\frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\therefore \; x_B = \begin{bmatrix} x_4 \\ x_1 \\ x_6 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \text{ and } E = 2$$

The net evaluations are given by

$$x_B = [C_B B^{-1} 1] = \begin{bmatrix} A \\ -C \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \\ -2 & 3 & -1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 4 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

Since there is negative not evaluation, the BFS obtained is not optimal. Here the only negative net evaluation is $z_3 - c_3 = -\frac{1}{2}$. So, x_3 is the next incoming basic variable.

Now we compute

$$y_3 - B^{-1}\hat{a}_3 = \begin{bmatrix} 1 & -\frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{3}{4} \\ -\frac{1}{2} \end{bmatrix}$$

These results are shown in the following simplex table

Basic variables	Values	$\hat{B}^{-1}$				$\hat{y}^{-1}$	min ratio
x_4	6	1	$-\frac{3}{4}$	0	0	$\frac{1}{4}$	12
x_1	1	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$	4
x_6	2	0	$-\frac{1}{4}$	1	0	$\frac{3}{4}$	$\frac{8}{3}$
z	2	0	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	

Here the minimum ratio is $\frac{8}{3}$ and is associated with the basic variable x_6. Therefore, the outgoing basic variable is x_6. So x_6 is replaced by x_3 is the next iteration. Using elementary row operations $\hat{y}_3 = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{3}{4} \\ -\frac{1}{2} \end{bmatrix}$ is converted to $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and the same operations are performed in $\hat{B}^{-1}$. This gives the new $\hat{B}^{-1}$ as follows

$$\text{Now } \hat{B}^{-1} = \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & \frac{4}{3} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix}$$

The next BFS is given by

$$\hat{x}_B = \begin{bmatrix} x_4 \\ x_1 \\ x_3 \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & \frac{4}{3} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ \frac{1}{3} \\ \frac{8}{3} \\ \frac{10}{3} \end{bmatrix}$$

$$\therefore \; x_B = \begin{bmatrix} x_4 \\ x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ \frac{1}{3} \\ \frac{8}{3} \end{bmatrix} \text{ and } z = \frac{10}{3}$$

The net evaluation are given by

$$[C_B B^{-1}]\begin{bmatrix} A \\ -c \end{bmatrix} = \left[0 \; \frac{1}{3} \; \frac{2}{3} \; 1\right] \begin{bmatrix} 3 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \\ -2 & 3 & -1 & 0 & 0 & 0 \end{bmatrix} = \left[0 \; 3 \; 0 \; 0 \; \frac{1}{3} \; \frac{2}{3}\right]$$

Here all net evaluations are non-negative. Hence we have obtained the optimal solution. The optimal solution is $x_1 = \frac{1}{3}, x_2 = 0, \frac{x}{3} = \frac{8}{3}$ and $z = \frac{10}{3}$.

Example 2.6.2. Solve by revised simplex method

Maximize $z = 5x_1 + 3x_2$

subject to $4x_1 + 5x_2 \leq 10$

$5x_1 + 2x_2 \leq 10$

$3x_1 + 8x_2 \leq 12$

$x_1, x_2 \geq 0$

Solution : Introducing surplus variable $x_3 \geq 0$, slack variables $x_4 \geq 0$, $x_5 \geq 0$ and artificial variable $x_6 \geq 0$ the standard form of the given LPP is

Maximize $z = 5x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5 - Mx_6$

subject to

$$4x_1 + 5x_2 - x_3 + x_6 = 10$$

$$5x_1 + 2x_2 + x_4 = 10$$

$$3x_1 + 8x_2 + x_5 = 12$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

or, Maximize $z = cx$

subject to $Ax = b$, $x \geq 0$

or, Maxmize $z = cx$

subject to $Ax = b$, $x \geq 0$

$$\text{where } A = \begin{bmatrix} 4 & 5 & -1 & 0 & 0 & 1 \\ 5 & 2 & 0 & 1 & 0 & 0 \\ 3 & 8 & 0 & 0 & 1 & 0 \end{bmatrix}, c = [5\ 3\ 0\ 0\ 0\ -M]$$

$$b = \begin{bmatrix} 10 \\ 10 \\ 12 \end{bmatrix}, x = [x_1\ x_2\ x_3\ x_4\ x_5\ x_6]^T$$

$$\therefore \text{ We have } A = \begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} 4 & 5 & -1 & 0 & 0 & 1 \\ 5 & 2 & 0 & 1 & 0 & 0 \\ 3 & 8 & 0 & 0 & 1 & 0 \\ -5 & -3 & 0 & 0 & 0 & M \end{bmatrix}, b = \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 12 \\ 0 \end{bmatrix}$$

$$\text{Initially, } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, x_B = \begin{bmatrix} x_6 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 12 \end{bmatrix}, C_B = [c_6\ c_4\ c_5] = [-M\ 0\ 0]$$

$$\therefore B^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Now } C_B B^{-1} = [-M\ 0\ 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [-M\ 0\ 0]$$

$$\therefore\ \hat{B}^{-1} = \begin{bmatrix} B^{-1} & 0 \\ C_B B^{-1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -M & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore\ \hat{x}_{B} = \begin{bmatrix} x_B \\ z \end{bmatrix} = \begin{bmatrix} x_6 \\ x_4 \\ x_5 \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -M & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 12 \\ -10M \end{bmatrix}$$

$$\therefore\ \begin{bmatrix} x_6 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 12 \end{bmatrix} \text{ and } z = -10M$$

The net evaluations are the components of

$$[c_B B^{-1}]\begin{bmatrix} A \\ -c \end{bmatrix} = [-M\ 0\ 0\ 1] \begin{bmatrix} 4 & 5 & -1 & 0 & 0 & 1 \\ 5 & 2 & 0 & 1 & 0 & 0 \\ 3 & 8 & 0 & 1 & 0 & 0 \\ -5 & -3 & 0 & 0 & 0 & M \end{bmatrix}$$

$$= [\,-4M-5\ \ -5M-3\ \ M\ \ 0\ \ 0\ \ 0\,]$$

$$= [\,z_1-c_1\ \ z_2-c_2\ \ z_3-c_3\ \ z_4-c_4\ \ z_5-c_5\ \ z_6-c_6\,]$$

Since there are negative net evaluations, the solution obtained is not optimal. The most negative net evaluation is $z_2 - c_2 = -5M - 3$. Therefore x_2 will be the new basic variable.

Now we compute

$$\hat{y}_2 = \hat{B}^{-1} a_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -M & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 8 \\ -5M-3 \end{bmatrix}$$

These results are shown in the following initial revised simplex table

Basic variables	Values	$\hat{B}^{-1}$				$\hat{y}^{-1}$	min ratio
x_6	10	1	0	0	0	5	2
x_4	10	0	1	0	0	2	5
x_5	10	0	0	1	0	8	$\frac{3}{2}$
z	– 10M	– M	0	0	1	– 5M – 3	

Here the minimum ratio is min $\left\{\frac{x_{Bi}}{y_{ik}} : y_{ik} > 0\right\} = \frac{3}{2}$ and the corresponding variable is x_5. Therefore, the outgoing basic variable is x_5. So x_5 is replaced by x_2 in the next table.

Using elementary row operations $\hat{y}_2 - \begin{bmatrix} 5 \\ 2 \\ 8 \\ -5M-3 \end{bmatrix}$ is converted to $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and the same operations are done for $\hat{B}^{-1}$. This gives new $\hat{B}^{-1}$ as follows

$$\hat{B}^{-1} = \begin{bmatrix} 1 & 0 & -\frac{5}{8} & 0 \\ 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{8} & 0 \\ -M & 0 & \frac{5M+3}{8} & 1 \end{bmatrix}$$

The new BFS is given by

$$x_B = \begin{bmatrix} x_6 \\ x_4 \\ x_2 \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} 1 & 0 & -\frac{5}{8} & 0 \\ 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{8} & 0 \\ -M & 0 & \frac{5M+3}{8} & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ 7 \\ \frac{3}{2} \\ \frac{-5M+9}{8} \end{bmatrix}$$

$$\therefore\ x_B = \begin{bmatrix} x_6 \\ x_4 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ 7 \\ \frac{3}{2} \end{bmatrix} \text{ and } z = \frac{-5M+9}{8}$$

The net evaluation are the componentts of

$$[c_B\, B^{-1}] \begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} -M & 0 & \frac{5M+3}{8} \end{bmatrix} \begin{bmatrix} 4 & 5 & -1 & 0 & 0 & 1 \\ 5 & 2 & 0 & 1 & 0 & 0 \\ 3 & 8 & 0 & 0 & 1 & 0 \\ -5 & -3 & 0 & 0 & 0 & M \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-17M-3}{8} & 0 & M & 0 & \frac{5M+3}{8} & 0 \end{bmatrix}$$

Since there is negative net evaluation, the BFS obtained is not optimal. Here the only negative net evaluation is $z_1 - c_1$. So x_1 is the next incoming basic variable.

Now we compute.

$$\hat{y}_1 + \hat{B}^{-1} a_1 = \begin{bmatrix} 1 & 0 & -\frac{5}{4} & 0 \\ 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{8} & 0 \\ -M & 0 & \frac{5M+3}{8} & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} \frac{17}{8} \\ \frac{17}{4} \\ \frac{3}{8} \\ \frac{-17M-31}{8} \end{bmatrix}$$

These results are shown in the following revised simplex table

Basic variables	Values	$\hat{B}^{-1}$				$\hat{y}^{-1}$	min ratio
x_6	$\frac{5}{2}$	1	0	$-\frac{5}{8}$	0	$\frac{17}{8}$	2
x_4	7	0	1	$-\frac{1}{4}$	0	$\frac{17}{4}$	5
x_2	$\frac{3}{2}$	0	0	$\frac{1}{8}$	0	$\frac{3}{8}$	$\frac{3}{2}$
z	$\frac{-5M+9}{8}$	$-M$	0	$\frac{5M+3}{8}$	1	$\frac{-17M-31}{8}$	

Here the minimum ratio is $\frac{20}{17}$ and is associated with the basic variable x_6. So x_6 is replaced by x_1 is the next iteration. Using elementary row operation $\hat{y}_1$ is converted to $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and the same operaions are performed in $\hat{B}^{-1}$ as follows

$$\therefore \text{ Now } \hat{B}^{-1} = \begin{bmatrix} \frac{8}{17} & 0 & -\frac{5}{17} & 0 \\ -2 & 1 & 1 & 0 \\ -\frac{3}{17} & 0 & \frac{4}{17} & 0 \\ \frac{31}{17} & - & -\frac{13}{17} & 1 \end{bmatrix}$$

The next BFS is given by

$$\hat{x}_B = \begin{bmatrix} x_1 \\ x_4 \\ x_2 \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} \frac{8}{17} & 0 & -\frac{5}{17} & 0 \\ -2 & 1 & 1 & 0 \\ -\frac{3}{17} & 0 & \frac{4}{17} & 0 \\ \frac{31}{17} & 0 & -\frac{13}{17} & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{20}{17} \\ 2 \\ \frac{18}{17} \\ \frac{154}{17} \end{bmatrix}$$

$$\therefore\ x_B = \begin{bmatrix} x_1 \\ x_4 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{20}{17} \\ 2 \\ \frac{18}{17} \end{bmatrix} \text{ and } z = \frac{154}{17}$$

The net evaluation are given by

$$\left[c_B\, B^{-1}\ 1\right]\begin{bmatrix} A \\ -c \end{bmatrix} = \left[\frac{31}{17}\ 0\ -\frac{13}{17}\ 1\right] \begin{bmatrix} 4 & 5 & -1 & 0 & 0 & 1 \\ 5 & 2 & 0 & 1 & 0 & 0 \\ 3 & 8 & 0 & 0 & 1 & 0 \\ -5 & -3 & 0 & 0 & 0 & M \end{bmatrix}$$

$$= \left[0\ 0\ -\frac{31}{17}\ 0\ -\frac{13}{17}\ \frac{31}{17} + M\right]$$

Since there are negative net evaluation the BFS obtained is not optimal. The most negative $z_j - c_j$ is $z_3 - c_3 = -\frac{31}{17}$ so x_3 is the next incoming basic variable.

Now we compute

$$\hat{y}_3 = \hat{B}^{-1}\hat{a}_3 = \begin{bmatrix} \frac{8}{17} & 0 & -\frac{5}{17} & 0 \\ -2 & 1 & 1 & 0 \\ -\frac{3}{17} & 0 & \frac{4}{17} & 0 \\ \frac{31}{17} & 0 & -\frac{13}{17} & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{8}{17} \\ 2 \\ \frac{3}{17} \\ -\frac{31}{17} \end{bmatrix}$$

These results are shown in the following revised simplex table.

Basic variables	Values	$\hat{B}^{-1}$				$\hat{y}^{-1}$	min ratio
x_1	$\frac{20}{17}$	$\frac{8}{17}$	0	$-\frac{5}{17}$	0	$-\frac{8}{17}$	...
x_4	2	-2	1	1	0	2	1
x_2	$\frac{18}{17}$	$-\frac{3}{17}$	0	$\frac{4}{17}$	0	$\frac{3}{17}$	6
z	$\frac{154}{17}$	$\frac{31}{17}$	0		1	$-\frac{38}{17}$	

Obviously x_4 will be replaced by x_3.

Using elementary row operations $\hat{y}_3$ is converted to $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and the same operations are used on $\hat{B}^{-1}$. This gives new $\hat{B}^{-1}$ as follows.

$$\text{New } \hat{B}^{-1} = \begin{bmatrix} 0 & \frac{4}{17} & -\frac{1}{17} & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{3}{34} & \frac{5}{34} & 0 \\ 0 & \frac{31}{34} & \frac{5}{34} & 1 \end{bmatrix}$$

The next BFS is given by

$$\hat{x}_B = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} 0 & \frac{4}{17} & -\frac{1}{17} & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{3}{34} & \frac{5}{34} & 0 \\ 0 & \frac{31}{34} & \frac{5}{34} & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{28}{17} \\ 1 \\ \frac{15}{17} \\ \frac{185}{17} \end{bmatrix}$$

$$\therefore\ x_B = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{28}{17} \\ 1 \\ \frac{15}{17} \end{bmatrix} \text{ and } z = \frac{185}{17}$$

The net evaluations are given by

$$[c_B B^{-1}]\begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} 0 & \frac{31}{34} & \frac{5}{34} & 1 \end{bmatrix} \begin{bmatrix} 4 & 5 & -1 & 0 & 0 & 1 \\ 5 & 2 & 0 & 1 & 0 & 0 \\ 3 & 8 & 0 & 0 & 1 & 0 \\ -5 & -3 & 0 & 0 & 0 & M \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \frac{31}{34} & \frac{5}{34} & M \end{bmatrix}$$

Here all net evaluation are found to be non-negative. Hence wehave obtained the optimal solution. The optimal solutions is given by

$$x_1 = \frac{28}{17},\ x_2 = \frac{15}{17} \text{ and } z_{max} = \frac{185}{17}.$$

2.7 Summary :

Revised simplex method is an efficient method and is very useful for large problem. Only necessary part of the simplex table is calculated to pass from one table to the next table. Standard form of the revised simplex method is devised and computational procedure of revised simplex method is noted and is compared with simplex method. Finally, the method is used to solve some examples.

2.8 Self Assessment Questions :

Use revised simplex method to solve the following LPP

1. Maximize $z = 3x_1 + 5x_2$

subject to $x_1 \le 4$

$x_2 \le 6$

$3x_1 + 2x_2 \le 18$

$x_1, x_2 \ge 0$

[Ans : $x_1 = 2$, $x_2 = 6$, $z_{max} = 36$]

2. Maximize $z = 6x_1 - 2x_2 + 3x_3$

subject to $2x_1 - x_2 + 2x_3 \le 2$

$x_1 \quad + 4x_3 \le 4$

$x_1, x_2, x_3 \ge 0$

[Ans : $x_1 = 4$, $x_2 = 6$, $x_3 = 0$ $z_{max} = 12$]

3. Maximize $z = x_1 + x_2$

subject to $x_1 + 2x_2 \ge 7$

$4x_1 + x_2 \ge 6$

$x_1, x_2 \ge 0$

$\left[\text{Ans}: x_1 = \frac{5}{7}, x_2 = \frac{22}{7}, z_{min} = \frac{27}{7}\right]$

4. Maxmize $z = 2x_1 + x_2$

subject to $3x_1 + x_2 \le 3$

$$4x_1 + 3x_2 \geq 6$$

$$x_1 + 2x_2 \leq 3$$

$$x_1, x_2 \geq 0$$

$$\left[\text{Ans}: x_1 = \frac{3}{5}, x_2 = \frac{6}{5}, z_{\min} = \frac{12}{5}\right]$$

5. Minimize $z = 4x_1 + 3x_2$

subject to

$$3x_1 + 4x_2 \leq 12$$

$$3x_1 + 3x_2 \leq 10$$

$$2x_1 + x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

$$\left[\text{Ans}: x_1 = \frac{4}{5}, x_2 = \frac{12}{5}, z_{\max} = \frac{52}{5}\right].$$

Unit 3 ❑ Dual Simplex Method

Structure

3.1 Introduction :

The Dual Simplex Method gives an algorithm in which we start with a basic optimal solution of the primal in which all $zj - ej \geq 0$ but not feasible is some basic solution are negative. At each iteration the number of negative basic variables are decreased while maintaining the optimality. An optimal solutionis reached in a finite number of steps. The benifit of this procedure lies in the fact that we need not take the help of any artificial variable and hence it reduces a lot of labour.

3.2 Comparison Between Simplex Method and Dual Simplex Method :

In simplex method the initial solution is basic feasible and non optimal. In subsequent tables the value of the objective function gradually increases and finally reaches to its obtimal value. In each table the solution is basic feasible and non-optimal.

In dual simplex method the initial solution is basic non-feasible and optimal. In subsequent tables the value of the objective function gradually decreases and finally reaches to its optimal value. In each table the solution is basic non-feasible and optimal (or better than optimal).

Basic non-feasible and better than optimal some $x_B < 0$ and all $zj - cj \geq 0$, $z_B \geq z_{max}$.

↓ (Dual simplex method) z gradually decreases

Basic feasible and optimal
$x_B \geq 0$ and all $zj - cj \geq 0$
$z_B = z_{max}$

↓ (Simplex method) z gradually increases

Basic feasible and non optimal
$x_B \geq 0$ and some $zj - cj < 0$, $z_B \leq z_{max}$

3.3 Applications of the Dual Simplex Method :

It the given LPP is optimal and infeasible then only dual simplex method is applicable for many practical problem the initial table does not satisty these conditions and as a consequence dual simplex method can not be applied. Simplex method has no such restnction and is applicable to any LPP. Hence as rule the regular simplexmethod preferred over the dual simplex method for solving the general LPP. However, there are instances when the dual simplex method has a distinct advantage over the regular simplex method. There are problems in which a dual feasible table is readily available to start the dual simplex method and for such problems the optimal BFS is obtained easily in comparison to simplex method. Some of the applications of dual simplex method are :

(i) Sensitivity analysis when the right hand side vector be changed or when new constraints are added.

(ii) Parametric programming.

(iii) Integer programming problem.

(iv) Some non-linear programming problem.

3.4 Criteria for Incoming and Outgoing basic Variable in Dual Simplex Method :

In the transmafion formula of simplex method if the basic variable xB_r is replaced by the non-basic variable x_k then we have

$$\hat{z} = z - \frac{xB_r}{y_{r_k}}(z_k - c_k) \qquad \dots (1)$$

and $$(\hat{z}_i - \hat{c}_j) - (z_j - c_i) - \frac{z_{rj}}{y_{rk}}(z_k - c_k) \qquad \dots (2)$$

As we want to remove negative besic variables, we choose the most negative besic variable xB_r (say) as outgoing basic variable from the list of all basic variables.

We know that if for some basic solution (not necessarily feasible) all components of net evaluations are non-negative then the value of the objective function to this basic solution is optimal or better than optimal. So we intend to lower down the value of the objective function to get z_{max}. For this from (1) we should have $y_{rk} < 0$ as $xB_r < 0$ and $z_k - c_k \geq 0$. This should be one criterion for incoming basic variable.

In the next table we want the solution to be optimal or better than optimal. So we should have $\hat{z}_j - \hat{c}_j \geq 0$ for all j.

∴ From (2) We have

$$z_j - c_j - \frac{y_{rj}}{y_{rk}}(z_k - c_k) \geq 0 \text{ for all } j$$

or, $$z_j - c_j \geq \frac{y_{rj}}{y_{rk}}(z_k - c_k) \text{ for all } j \qquad \dots (3)$$

When $y\bar{j} \geq 0$ then (3) is satisfied as $y_{rk} < 0$ and all $zj - cj \geq 0$.

When $y_r j < 0$ then (3) is satisfied if $\frac{z_j - c_j}{y_{rj}} \leq \frac{z_k - c_k}{y_{rk}}$ for all j

Hence we are b choose k such that

$$\max_{y_{rj} < 0} \left\{ \frac{z_j - c_j}{y_{rj}} \right\} = \frac{z_k - c_k}{y_{rk}}$$

3.5 Dual Simplex Algorithm :

The iterative procedure for dual simplex algorithm are as follows :

Step 1 : Convert the minimizatism LPP into that of maximization if it is in the minimization form.

Step 2 : Convert the ≥ type inequalious, representing the constraints of the given LPP, if any, into those of ≤ type by multiplying the corresponding constraints by –1.

Step 3 : Introduce slack variables in the constraints of the given LPP and obtain an initial basic solution. Put this solution in the starting dual simplex table.

Step 4 : Test the nature of the net evaluations $zj - cj$ in the starting simplex table.

(i) If all $zj - ej$ and xBj are non negative for all i and j, then an optimum basic fesible solution has been obtained.

(ii) It all $zj - cj$ are non negative and at least one basic variable, say xB_r, is negative then go to step 5.

(iii) It at least one $zj - cj$ is negative then dual simplex method is not applicable. In this case we are to apply artificial constraint method.

Step 5 : Select the most negative basic variable, say xB_k, as outgoing basic variable.

Step 6 : Test the nature of all $y_r j$, $j = 1, z, \ldots, n$.

(i) It all $y_r j$ are non-negative, there does not exist any feasible solution to the given LPP.

(ii) It at least one $y_k j$ is negative, then comute

$$\left\{\frac{zj}{y_{kj}} : y_{kj} < 0\right\}, j = 1, 2, \ldots, n,$$

and choose the maximum of these, If the maximum of these be $\frac{z_r - c_r}{y_{kr}}$ then x_r is the incoming basic variable i.e. xB_k is replaced by x_r.

Step 7 : With y_{kr} as the key element form the next table. Using elementary row operation convert the key element to unity and all other elements of the key column to zero to get the improoed solution.

Step 8 : Repeat the steps 4 to 7 until either an optimun basic feasible solution is obtained or there is an indication of no feasible solution.

3.6 Illustrative Examples :

Example 3.6.1. Solve the following LPP by dual Simplex Method.

Maximize $z = 2x_1 + x_2$

Subject to $3x_1 + x_2 \geq 3$

$4x_1 + x_2 \geq 6$

$x_1 + 2x_2 \geq 3$

$x_1, x_2 \geq 0.$

Solution : Converting the given LPP into maximization and changing all ≥ type inequations to ≤ type and finally adding slack variables $x_3 \geq 0$, $x_4 \geq 0$, $x_5 \geq 0$, the reformulated LPP in its standard form becomes.

Maximize $z' = -2x_1 - x_2 + ox_3 + ox_4 + ox_5$

Subject to $-3x_1 - x_2 + x_3 = 3$

$-4x_1 - x_2 + x_4 = 6$

$-x_1 - 2x_2 + x_5 = 3$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$

Solution : The solution of this LPP by dual simplex method is shown in the following tables.

		c_j	-2	-1	0	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	
0	y_3	-3	-3	-1	1	0	0	
0	y_4	-6	$\boxed{-4}$	-1	0	1	0	$\rightarrow$
0	y_5	-3	-1	-2	0	0	1	
$z' = 0$		$zj - cj$	2	1	0	0	0	
$\frac{z_j - c_j}{y_{2j}} : y_{2j} < 0$			$\frac{2}{-4}$	$\frac{1}{-1}$				
0	$\frac{y}{3}$	$\frac{3}{2}$	0	$-\frac{1}{4}$	1	$-\frac{3}{4}$	0	
-2	y_1	$\frac{3}{2}$	1	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	
0	y_5	$-\frac{3}{2}$	0	$\boxed{-\frac{7}{4}}$	0	$-\frac{1}{4}$	0	$\rightarrow$
$z' = -3$		$zj - cj$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	
$\frac{z_j - c_j}{y_{3j}} : y_{3j} < 0$			$\frac{1}{2}\Big/\left(-\frac{7}{4}\right) = 2\frac{2}{7}$					
0	y_3	$\frac{12}{7}$	0	0	1	$-\frac{5}{7}$	$-\frac{1}{7}$	
-2	y_1	$\frac{9}{7}$	1	0	0	$-\frac{2}{7}$	$\frac{1}{7}$	
-1	y_2	$\frac{6}{7}$	0	1	0	$\frac{1}{7}$	$\frac{4}{7}$	
$z' = -\frac{24}{7}$	$zj - cj$	0	0	0	0	$\frac{3}{7}$	$\frac{2}{7}$	

In the first table max $\left\{-\frac{1}{2}, -1\right\} = -\frac{1}{2}$ and is ascociated with y_1 $\therefore$ y_4 is replaced by y_1 for the second table.

In the second table there is only one ratio $-\frac{2}{7}$ and is associated with y_2.

$\therefore$ y_5 is replaced by y_2 for the third table.

In the third table all xB_i are non negative. So this is optimal table. The optimal solution is $x_1 = \frac{9}{7}$, $x_2 = \frac{6}{7}$ and $z'_{max} = -\frac{24}{7}$ $\therefore$ $z_{min} = -z'_{max} = \frac{24}{7}$.

Example 3.6.2. Solve the following LPP by dual Simplex Method.

$$\begin{aligned} \textbf{Maximize} \quad & z = -2x_1 - 2x_2 - 4x_3 \\ \textbf{Subject to} \quad & 2x_1 + 3x_2 + 5x_3 \le 2 \\ & 3x_1 + x_2 + 2x_3 \ge 3 \\ & x_1 + 4x_2 + 6x_3 \ge 5 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

Solution : Converting the $\ge$ type inequations into $\le$ type and introducing the slack variables $x_4 \ge 0$, $x_5 \ge 0$, $x_6 \ge 0$ the given LPP can be written in the standard form as

$$\begin{aligned} \textbf{Maximize} \quad & z = -2x_1 - 2x_2 - 4x_3 + 0x_4 + 0x_5 + 0x_6 \\ \textbf{Subject to} \quad & 2x_1 + 3x_2 + 5x_3 + x_4 = 2 \\ & -3x_1 - x_2 - 2x_3 + x_5 = -3 \\ & -x_1 - 4x_2 - 6x_3 + x_6 = -5 \\ & x_1, x_2, x_3, x_4, x_6 \ge 0. \end{aligned}$$

The following tables are obtained by using dual simplex method to this LPP.

		c_j	-2	-2	-4	0	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	
0	y_4	2	2	3	5	1	0	0	
0	y_5	-3	-3	-1	-2	0	1	0	
0	y_6	-5	-1	$\boxed{-4}$	-6	0	0	1	→
$z = 0$		$z_j - c_j$	2	2	4	0	0	0	
$\frac{z_j - c_i}{y_{3j}} : y_{3j} < 0$			$\frac{2}{(-1)} = -2$	$\frac{2}{(-4)} = -\frac{1}{2}$	$\frac{4}{(-6)} = -\frac{2}{3}$				
0	y_4	$-\frac{7}{4}$	$\frac{5}{4}$	0	$\frac{1}{2}$	1	0	$\frac{3}{4}$	
0	x_5	$-\frac{7}{4}$	$\boxed{-\frac{11}{4}}$	0	$-\frac{1}{2}$	0	1	$-\frac{1}{4}$	
-2	y_2	$\frac{5}{4}$	$\frac{1}{4}$	1	$\frac{3}{2}$	0	0	$-\frac{1}{4}$	
$z = -\frac{5}{2}$		$z_j - c_j$	$\frac{3}{2}$	0	1	0	0	$\frac{1}{2}$	
$\frac{z_j - c_i}{y_{2j}} : y_{2j} < 0$			$\frac{\left(\frac{3}{2}\right)}{\left(-\frac{11}{4}\right)} = -\frac{6}{11}$		$\frac{1}{\left(-\frac{1}{2}\right)} = -2$			$\frac{\left(\frac{1}{2}\right)}{\left(-\frac{1}{4}\right)} = -2$	
0	y_4	$-\frac{28}{11}$	0	0	$\frac{3}{11}$	1	$\frac{5}{11}$	$\frac{7}{11}$	
-2	y_1	$\frac{7}{11}$	1	0	$\frac{2}{11}$	0	$-\frac{4}{11}$	$\frac{1}{11}$	
-2	y_2	$\frac{12}{11}$	0	1	$\frac{16}{11}$	0	$\frac{1}{11}$	$-\frac{3}{11}$	
$z = -\frac{38}{11}$		$z_j - c_j$	0	0	$\frac{8}{11}$	0	$\frac{6}{11}$	$\frac{4}{11}$	

In the first table $x_6 = -5$ is the most negative basic variable and max $\left\{-2, -\frac{1}{2}, -\frac{2}{3}\right\} = -\frac{1}{2}$ which is associated with this non basic variable x_2. So x_6 is replaced by x_2.

In the second table $x_4 = -\frac{7}{4}$, $x_5 = -\frac{7}{4}$ are the most negative basic variables. We choose x_5 arbitrarily. Here max $\left\{-\frac{6}{11}, -2, -2\right\} = -\frac{6}{11}$ which is associated with the non basic variable x_1. So x_5 is replaced by x_1.

In the third table $xB_1 = x_4 < 0$ and all $y_1 j \geq 0$.

$\therefore$ The given LPP has no feasible solution.

3.7 Modification of Dual Simplex Method :

If the initial table of the dual simplex method contains some negative basic variables and some of the net-evaluations are negtive then the dual simplex method is not applicable. In such situation dual simplex method is to be modified to from an equivalent LPP in which some basic variables are negative but all netevaluations are non-negative. Hence standard dual simplex method can be applied to that equivalent LPP.

The artificial constraint is one such method. In this method we consider the variables corresponding to which the net evaluations are negative and the variable corresponding to the most negative component of net evaluations is noted. Lt $z_p - c_p$ be the most negative net evalution. So we consider the corresponding variable x_p. In this method we have to consider the artificial constraint.

$$\Sigma x_j \leq M$$

Where Σ is extended over all j's for which $zj - cj < 0$ and M is a sufficiently large positive number. Adding slack variable x_M to this constraint we get

$$\Sigma xj + x_M = M$$

From this we find x_p as

$$x_p = M - \left(x_M + \sum_{j \neq p} x_j\right)$$

This x_p is then subptituted in the orginal objective function and in the set of all constraints. This new problem together with the new added artificial constraint is equivalent to the given problem. This equivalent LPP will have all $zj - cj \geq 0$. Thus dual samplex method can be applied.

3.8 Illustrative Examples :

Example 3.8.1 : Use the artificial constraint method to find the initial basic solution of the following problem and then apply the dual simplex algorithm to solve it

Maximize $z = -2x_1 - x_2 - x_3$

Subject to $4x_1 + 6x_2 + 3x_3 \leq 8$

$-x_1 + 9x_2 - x_3 \geq 3$

$2x_1 + 3x_2 - 5x_3 \geq 4$

$x_1, x_2, x_3 \geq 0$

Solution : We first convert the minimization problem to maximization and then change the inequation of $\geq$ type into $\leq$ type. Finally adding slack variables $x_4 \geq 0$, $x_5 \geq 0$, $x_6 \geq 0$ we get the standard form LPP in dual simplex method as

Maximize $z' = 2x_1 + x_2 + x_3 + ox_4 + ox_5 + ox_6$

Subject to $4x_1 + 6x_2 + 3x_3 + x_4 = 8$

$x_1 - 9x_2 + x_3 + x_5 = -3$

$-2x_1 - 3x_2 + 5x_3 + x_6 = -4$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$

The initial dual simplex table is

		c_j	2	1	1	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_4	8	4	6	3	1	0	0
0	y_5	–3	1	–9	1	0	1	0
0	y_6	–4	–2	–3	5	0	0	1
$z' = 0$		$z_j - c_j$	–2	–1	–1	0	0	0

Here there are negative net evaluation, so standard dual simplex method is not applicable.

The negative net evaluations are $z_1 - c_1, z_2 - c_2, z_3 - c_3$ & most negative net evaluation is $z_1 - c_1 = -2$.

$\therefore$ The artificial constraint is

$x_1 + x_2 + x_3 \leq M$ where M is a very large positive number. Adding slack variable x_M we have

$$x_1 + x_2 + x_3 + x_M = M$$

From this we have $x_1 = M - x_2 - x_3 - x_M$

Using this in the LPP and adding the artificial constraint we have.

Maximize $z' = 2(M - x_2 - x_3 - x_M) + x_2 + x_3 + ox_4 + ox_5 + ox_6$

Subject to $4(M - x_2 - x_3 - x_M) + 6x_2 + 3x_3 + x_4 = 8$

$(M - x_2 - x_3 - x_M) - 9x_2 + x_3 + x_5 = -3$

$-2(M - x_2 - x_3 - x_M) - 3x_2 + 5x_3 + x_6 = -4$

$x_1 + x_1 + x_3 + x_M = M$

$x_1, x_2, x_3, x_4, x_5, x_6, x_M \geq 0$

or, **Maximize** $z' = -2x_M - x_2 - x_3 + ox_4 + ox_5 + 2M$

Subject to $-4x_M + 2x_2 - x_3 + x_4 = 8 - 4M$

$-x_M - 10x_2 + x_5 = -3 - M$

$2x_M - x_2 + 7x_3 + x_6 = -4 + 2M$

$x_M + x_1 + x_2 + x_3 = M$

$x_1, x_2, x_3, x_4, x_5, x_6, x_M \geq 0$

The following tables are obtained using simplex method.

c_B	y_B	x_B	y_M	y_1	y_2	y_3	y_4	y_5	y_6	
		cj	-2	0	-1	-1	0	0	0	
0	y_4	$8-4M$	$\boxed{-4}$	0	2	-1	1	0	0	→
0	y_5	$-3-M$	-1	0	-10	0	0	1	0	
0	y_6	$-4+2M$	2	0	-1	7	0	0	1	
0	y_1	M	1	1	1	1	0	0	0	
$z' = 0$		$zj - cj$	2	1	1	1	0	0	0	
$\frac{z_j - c_j}{y_{1j}} : y_{1j} < 0$			$-\frac{1}{2}$	-1						
-2	y_M	$-2+M$	1	0	$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	0	0	
0	y_5	-5	0	0	$\boxed{-\frac{21}{2}}$	$\frac{1}{4}$	$-\frac{1}{4}$	1	0	
0	y_6	0	0	0	0	$\frac{13}{2}$	$\frac{1}{2}$	0	0	
0	y_1	2	0	1	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	0	0	
$z' = 4 - 2M$		$zj - c_i$	0	0	2	$\frac{1}{2}$	$\frac{1}{2}$	0	0	
$\frac{z_j - c_j}{y_{2j}} : y_{2j} < 0$					$-\frac{4}{21}$		-2			
-2	y_M	$M - \frac{37}{21}$	1	0	0	$\frac{5}{21}$	$-\frac{5}{21}$	$-\frac{1}{21}$	0	
-1	y_2	$\frac{10}{21}$	0	0	1	$-\frac{1}{42}$	$\frac{1}{42}$	$-\frac{2}{21}$	0	
0	y_6	0	0	0	0	$\frac{13}{2}$	$\frac{1}{2}$	0	1	
0	y_1	$\frac{9}{7}$	0	1	0	$\frac{11}{14}$	$\frac{3}{14}$	$\frac{1}{7}$	0	
$z' = -2M + \frac{64}{21}$		$zj - cj$	0	0	0	$\frac{13}{42}$	$\frac{19}{42}$	$\frac{4}{21}$	0	

Here all basic variable are non-negative. So thes is the optimal table. The optimal solution is $x_1 = \frac{9}{7}$ $x_2 = \frac{10}{21}$, $x_3 = 0$ and $z'_{max} = \left(-2M + \frac{64}{21}\right) + 2M = \frac{64}{21}$ Therefore $z_{min} = -z'_{max} = -\frac{64}{21}$.

Example 3.8.2. Use the atificial constrant method to dind the initial basic solution of the following problem and then apply the dual simplex algorithm to solve it :

Maximize $z = 2x_1 - 3x_2 - 2x_3$

Subject to $x_1 - 2x_2 - 3x_3 = 8$

$2x_2 + x_3 \leq 10$

$x_2 - 2x_3 \geq 4$

$x_1, x_2, x_3 \geq 0.$

Solution : We first change the inequation of ≥ type into ≤ type. Adding slack variable $x_4 \geq 0$, $x_5 \geq 0$ we get the standard form of the LPP in dual simplex method as

Maximize $z = 2x_1 - 3x_2 - 2x_3$

Subject to $x_1 - 2x_2 - 3x_3 = 8$

$2x_2 + x_3 + x_4 = 10$

$-x_2 + 2x_3 + x5 = -4$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

The initial dual simplex table is

		cj	2	−3	−2	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
2	y_1	8	1	−2	−3	0	0
0	y_4	10	0	2	1	1	0
0	y_5	−4	0	−1	2	0	1
$z = 16$		$zj - cj$	0	−1	−4	0	0

Since these are negative net evaluations, standard dual simplex method is not applicable. The negative net evaluations are $z_2 - c_2$ and $z_3 - c_3$ and most negative net evaluation is $z_3 - c_3 = -4$.

$\therefore$ The artificial constraint is $x_2 + x_3 \leq M$ where M is a very large positive number. Adding plack variable x_M we have

$$x_2 + x_3 + x_M = M$$

From this we have $x_3 = M - x_2 - x_M$

Using this in the LPP and adding the artificial constraint we have the equivalent LPP as

Maximize $z = 2x_M + 2x_1 - x_2 - 2M$

Subject to

$$3x_M + x_1 + x_2 = 3M + 8a$$

$$-x_M + x_2 + x_4 = -M + 10$$

$$-2x_M - 3x_2 + x\% = -2M - 4$$

$$x_M + x_2 + x_3 = M$$

$$x_M, x_1, x_2, x_3, x_4, x_5 \geq 0.$$

The dual simplex tables are as follows.

		c_j	2	2	−1	0	0	0
c_B	x_B	x_B	y_M	y_1	y_2	y_3	y_4	y_5
2	y_1	$3M + 8$	3	1	1	0	0	0
0	y_4	$-M + 10$	−1	0	1	0	1	0
0	y_5	$-2M - 4$	−2	0	[−3]	0	0	1
0	z_3	M	1	0	1	1	0	0
$z = 6M + 16$		$z_j - c_j$	4	0	3	0	0	0
$\frac{z_j - c_j}{y_{3j}} : y_{3j} < 0$			−2		−1			
2	y_1	$\frac{7M+20}{3}$	$\frac{7}{3}$	1	0	0	0	$\frac{1}{3}$
0	y_4	$\frac{-5M+26}{3}$	[$-\frac{5}{3}$]	0	0	0	1	$\frac{1}{3}$
−1	y_2	$\frac{2M+4}{3}$	$\frac{2}{3}$	0	1	0	0	$-\frac{1}{3}$
0	y_3	$\frac{M-4}{3}$	$\frac{1}{3}$	0	0	1	0	$\frac{1}{3}$
$z = 4M + 12$		$z_j - c_j$	2	0	0	0	0	1
$\frac{z_j - c_j}{y_{2j}} : y_{2j} < 0$			$-\frac{6}{5}$					
2	y_1	$\frac{94}{5}$	0	1	0	0	$\frac{7}{5}$	$\frac{4}{5}$
2	y_M	$\frac{5M-26}{5}$	1	0	0	0	$-\frac{3}{5}$	$-\frac{1}{5}$
−1	y_2	$\frac{24}{5}$	0	0	1	0	$\frac{2}{5}$	$-\frac{1}{5}$
0	y_3	$\frac{2}{5}$	0	0	0	1	$\frac{1}{5}$	$\frac{2}{5}$
$z = 10M + 112$		$z_j - c_j$	0	0	0	0	$\frac{6}{5}$	$\frac{7}{5}$

In this table all basic variables are non-negative. So this is the optimal table. The optimal solution is $x_1 = \frac{94}{5}$, $x_2 = \frac{24}{5}$, $x_3 = \frac{2}{5}$ and $z_{max} = \frac{10M+112}{5} - 2M = \frac{112}{5}$.

3.9 Summary :

Dual simplex method is found to be very useful is a large class of LPP. It is simple to handle and sige of the tables are not large as no artificial variables are introduced, the method is illustrated through examples. The method is then modified to handle more LPP.

3.10 Self Assessment Questions :

1. Use dual simplex method to solve the LPP

   Maximize $z = -2x_1 - 3x_2 - x_3$

   Subject to $2x_1 + x_2 + 2x_3 \geq 3$

   $3x_1 + 2x_2 + x_3 \geq 4$

   $x_1, x_2, x_3 \geq 0$

   [Ans. $x_1 = \frac{5}{4}$, $x_2 = 0$, $x_3 = \frac{1}{4}$, $z_{max} = -\frac{11}{4}$]

2. Use dual simplex method to solve the LPP

   Maximize $z = 10x_1 + 6x_2 + 2x_3$

   Subject to $-x_1 + x_2 + x_3 \geq 1$

   $3x_1 + x_2 - x_3 \geq 2$

   $x_1, x_2, x_3 \geq 0$

   [Ans. $x_1 = \frac{1}{4}$, $x_2 = \frac{5}{4}$, $x_3 = 0$, $z_{min} = 10$]

3. Solve by dual simplex method the fllowing LPP

   Maximize $z = 6x_1 + x_2$

Subject to $\quad 2x_1 + x_2 \geq 3$

$x_1 + x_2 \geq 0$

$x_1, x_2 \geq 0$

[Ans. $x_1 = 1, x_2 = 1, z_{min} = 7$]

4. Solve the following LPP by dual simplex method

Maximize $z = -3x_1 - 2x_2$

Subject to $\quad x_1 + x_2 \geq 1$

$x_1 + x_2 \leq 7$

$x_1 + 2x_2 \geq 10$

$x_2 \leq 3$

$x_1, x_2 \geq 0$

[Ans. $x_1 = 4, x_2 = 3, z_{max} = -18$]

5. Solve by dual simplex method :

Maximize $z = 2x_1 + 3x_2$

Subject to $\quad 2x_1 + 3x_2 \leq 30$

$x_1 + 2x_2 \geq 10$

$x_1 - x_2 \geq 0$

$x_1 \geq 5$

$x_2 \geq 0$

[Ans. $x_1 = 5, x_2 = \frac{5}{2}, z_{min} = \frac{35}{2}$]

6. Solve the following LPP by daul simplex method

Maximize $z = x_1 + x_2$

Subject to $2x_1 + x_2 \geq 2$

$-x_1 - x_2 \geq 1$

$x_1, x_2 \geq 0$

[Ans. No feasible solution]

7. Using artificial constraint procedure, solve the follwing problem by dual simplex method and show that the problem has no feasible solution

Maximize $z = -x_1 + x_2$

Subject to $x_1 - 4x_2 \geq 5$

$x_1 - 3x_2 \leq 1$

$2x_1 - 5x_2 \geq 1$

$x_1, x_2 \geq 0$

8. Use the artificial constraint method to find the initial basic solution of the following problem and then apply the dual simplex algorthm to solve it

Maximize $z = x_1 - 3x_2 - 2x_3$

Subject to $x_2 - 2x_3 \geq 2$

$x_1 - 4x_2 - 6x_3 = 8$

$2x_2 + x_3 \leq 5$

$x_1, x_2, x_3 \geq 0$

[Ans. $x_1 = \frac{94}{5}$, $x_3 = \frac{12}{5}$, $x_3 = \frac{1}{5}$, $z_{max} = \frac{56}{5}$]

Unit 4 ❑ Post Optimality Analysis

Structure

4.1 Introduction :

In reality the problem accuring are in general large in size and often an error is discovered in the data after the attainment of an optimal solution to the problem. In such a situation there are two alternatives, either to solve the problem from begining or to device some method to use the optimal table. Undoubtedly the second one willsave time and space and is named as post optimality analysis. Also in practical situation the values of the co-efficient matrix A, the components of the requirement vector and the cost vector or neither known exactly nor they are constant for all time and or all situations. so it is important to know how sensitive the optimal solution is to small changes in these parameters. By sensitiveness we mean fulfilments of the condition of optimality as well as determining the limits of variations of these parameters for the solution to remain optimal.

We shall study the following effects of changes in the

(i) co-efficients c_j of the objective function.

(ii) components of the requirement vector to

(iii) addition of a new variable

(iv) deletion of a variable

(v) addition of a new constraint

4.2 Discrete Changes In The Cost Vector :

Let x_B be the optimal basic solution of the LPP

Maximize $z = cx$

Subject to $Ax = b$

$x \geq 0$

Where $c, x^T \in R^n$, $b^T \in R^m$ and A is mxn an real matrix. Let Δc_k be the amount by which c_k is changed. So the new value of c_k is $c_k^* = c_k + \Delta c_k$.

We know that $x_B = B^{-1} b$ and so it independent of c.

As initially x_B was BFS it will remain so after the change. The optimality condition is $z_j - c_j \geq 0$ for all j i.e. $[c_B B^{-1} 1]\begin{bmatrix} A \\ -c \end{bmatrix} \geq 0$. It invalues c. So change in c will affect this condition. Thus when c_x is changed to c_k^*, the solution x_B may or may not remain optimal solution though it remains BFS.

Two cases will arise

(i) c_k is not in c_B

(ii) c_k is in c_B

Case (i). Here c_k is not in c_B. The net evaluations are the components of $c_B B^{-1}A - c_1$ and as $x_B = B^{-1}b$ was optimal solution we have $c_B B^{-1}A - c \geq 0$. i.e. $z_j - c_j \geq 0 \; \forall \; j$.

We note that when c_k is changed to c_k^* only kth component of net evaluation will change Thus for all $j = 1, 2, \ldots, k-1, k+1, \ldots, n$ i.e. for all $j \neq k$ we have new net evaluations.

$$z_j^* - c_j^* = z_j - c_j \geq 0 \text{ [as } z_j - c_j \geq 0 \text{ for all } j]$$

$$\text{for } j = k \text{ we have } z_k^* - c_k^* = z_k - (c_k + \Delta c_k) = (z_k - c_k) - \Delta c_k$$

We have $z_k - c_k \geq 0$. Therefor for all Δc_k, $z_k^* - c_k^*$ will not remain non negative.

Thus x_B will remain optimal solution for the changed LPP if $z_k - c_k - \Delta c_k \geq 0$ i.e. if $\Delta c_k \leq z_k - c_k$.

Case (ii). Here c_k is one component of c_B. Let $c_k = c_{B_\lambda}$ and so x_k is a basic variable. Thus y_k is a unit vector with its λth component as 1.

The new value of $z_k - c_k$ is given by

$$z_k^* - c_k^* = \sum_{\substack{i=1\\ i\neq\lambda}}^{m} c_{B_i} y_{i_k} + c_k^*.1 - c_k^*.1 = 0 \quad [\because y_{ik} = 0 \ \forall \ i \neq \lambda].$$

For $j \neq k$, new value of $z_j - c_j$ is given by

$$z_j^* - c_j^* = \left(\sum_{\substack{i=1\\ i\neq\lambda}}^{m} c_{B_i} y_{i_j} + c_k^*. y_{\lambda_j} \right) - c_j^*$$

$$= \sum_{i\neq\lambda} c_{Bi} y_{ij} + (c_k + \Delta c_k) y_{\lambda j} - c_j \left[\because c_j^* = c_j \ \forall \ j \neq k\right]$$

$$= \sum_{i\neq\lambda} c_{Bi} y_{ij} + c_{B_\lambda} y_{\lambda j} + \Delta c_k y_{\lambda j} - c_j) \ . \left[\because c_k = y_{B\lambda}\right]$$

$$= \sum_{i=1}^{m} c_{Bj} y_{ij} + \Delta c_k y_{\lambda j} - c_j$$

$$= z_j - c_j + \Delta c_k y_{\lambda i} \left[\because \ z_j = \sum_{i=1}^{m} c_{Bi} y_{ij}\right]$$

$\therefore$ x_B remains optimal solution

if $z_j - c_j + \Delta c_k y_{\lambda j} \geq 0 \ \forall \ i \neq k$

i.e. if $\Delta c_k \ y_{\lambda j} \geq -(z_j - c_j) \ \forall \ j \neq k$

Now for $y_{\lambda j} = 0$ this condition is fulfilled automatically as $z_j - c_j \geq 0$.

For $y_{\lambda j} > 0$ this condition is satisfied if $\Delta c_k \geq -\dfrac{z_j - c_i}{y_{\lambda j}} \ \forall j \neq k$

i.e. if $-\dfrac{z_j - c_i}{y_{\lambda j}} \leq \Delta c_k \forall j \neq k$

$\therefore$ We must have $\max\limits_{\substack{y_{\lambda j}>0 \\ j \neq k}} \left\{ -\dfrac{z_j - c_j}{y_{\lambda i}} \right\} \leq \Delta c_k$

$y_{\lambda j} < 0$ this condition is satistied if $\Delta c_k \leq -\dfrac{z_j - c_j}{y_{\lambda j}} \forall j \neq k$

$\therefore$ We must have $\Delta c_k \leq \min\limits_{\substack{y_{\lambda j}>0 \\ j \neq k}} \left\{ -\dfrac{z_j - c_j}{y_{\lambda j}} \right\}$

These two conditions can be combined as

$$\max_{\substack{y_{\lambda j}>0 \\ j \neq k}} \left\{ -\frac{z_j - c_j}{y_{\lambda j}} \right\} \leq \Delta c_k \leq \min_{\substack{y_{\lambda j}<0 \\ j \neq k}} \left\{ -\frac{z_j - c_j}{y_{\lambda j}} \right\}$$

Hence if Δc_k lies in this range then the solution x_B remain optimal and if Δc_k falls outsids this range then at least one $z_j - c_j$ will be negative and the solution will no longer remain optimal.

It no $y_{\lambda j} > 0$, then there is no lower bownd of Δc_k and if no $y_{\lambda j} < 0$, then there is no upper bound of Δc_k.

4.3 Illustrative Example :

4.3.1 The optimal solution of the LPP :

Maximize $z = 6x_1 - 2x_2 + 3x_3$

Subject to $2x_1 - x_2 + 2x_3 \le 2$

$x_1 \quad + 4x_3 \le 4$

$x_1, x_2, x_3 \ge 0.$

is contained in the table.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
6	y_1	4	1	0	4	0	1
–2	y_2	6	0	1	6	–1	2
$z_j - c_j$		$z = 12$	0	0	9	2	2

Find the ranges of the cost components when (i) changed one at a time (ii) changed two at a time (iii) changed all three at a time to keep the optimal solution same.

Solution :

(i) When one component is changed at a time :

For change of $c_1 = 6$ to c_1^* we have the corresponding changed table as

		c_j	c_1^*	–2	3	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
c_1^*	y_1	4	1	0	4	0	1
–2	y_2	6	0	1	6	–1	2
			0	0	$4c_1^* - 12$	2	$c_1^* - 4$

This table becomes optimal table

if $4c_1^* - 12 \ge 0$ and $c_1^* - 4 \ge 0$

i.e. if $c_1^* \ge 3$ and $c_1^* \ge 4$

i.e. if $c_1^* \ge 4$

For change of $c_2 = -2$ to c_2^* the table corresponding to the final table becomes.

			6	c_2^*	3	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
6	y_1	4	1	0	4	0	1
c_2^*	y_2	6	0	1	6	–1	2
			0	0	$24 + 6c_2^*$	$-c_2^*$	$6 + 2c_2^*$

This table becomes the optimal table

$y\ 24 + 2c_2^* \geq 0$ and $-c_2^* \geq 0$ and $6 + 2c_2^* \geq 0$

i.e. if $c_2^* \geq -4$ and $c_2^* \leq 0$ and $c_2^* \geq -3$

i.e. if $-3 \leq c_2^* \leq 0$

For change of $c_3 = 3$ to c_3^* the modified table is

			c	–2	c_3^*	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
6	y_1	4	1	0	4	0	1
–2	y_2	6	0	1	6	–1	2
			0	0	$12 + c_3^*$	2	2

This table remains optimal table

if $12 - c_3^* \geq 0$

i.e. if $c_3^* \leq 12$

(ii) When two components are changed at a time.

For the change of $c_1 = 6$ and $c_2 = -2$ to c_1^* and c_2^* the modified table is

			c_1^*	c_2^*	3	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
c_1^*	y_1	4	1	0	4	0	1
c_2^*	y_2	6	0	1	6	–1	2
			0	0	$4c_1^* + 6c_2^* - 3$	$-c_2^*$	$c_2^* + 2c_2^*$

This table becomes optimal table if all $z_j - c_j \geq 0$

i.e. if $4c_1^* + 6c_2^* - 3 \geq 0$ and $-c_2^* \geq 0$ and $c_1^* + 2c_2^* \geq 0$

i.e. if $c_2^* \geq \dfrac{3 - 4c_1^*}{6}$ and $c_2^* \leq 0$ and $c_2^* \geq -\dfrac{c_1^*}{2}$

i.e. $\max\left\{\dfrac{3 - 4c_1^*}{6}, \dfrac{-c_1^*}{2}\right\} \leq c_2^* \leq 0$ and c_1^* any real number.

For the change of $c_1 = 6$ and $c_3 = 3$ to c_1^* and c_3^* respectively the modified table is

			c_1^*	-2	c_3^*	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
c_1^*	y_1	4	1	0	4	0	1
-2	y_2	6	0	1	6	-1	2
			0	0	$4c_1^* - 12c_2^* - c_3^*$	2	$c_1^* - 4$

This table remains optimal table if all $z_j - c_j \geq 0$

i.e. if $4c_1^* - 12 - c_3^* \geq 0$ and $c_1^* - 4 \geq 0$

i.e. if $c_1^* \geq \dfrac{12 + c_3^*}{4}$ and $c_1^* \geq 4$

i.e. if $c_1^* \geq \max\left\{4, 3 + \dfrac{c_3^*}{4}\right\}$ and c_3^* any real number.

For the change of $c_2 = -2$ and $c_3 = 3$ to c_2^* and c_3^* respectively the modified table is

			6	c_2^*	c_3^*	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
6	y_1	4	1	0	4	0	1
c_2^*	y_2	6	0	1	6	-1	2
			0	0	$24 + 6c_2^* - c_3^*$	$-c_2^*$	$6 + 2c_2^*$

This table remains optimal table if all $zj - cj \geq 0$

i.e. if $24 + 6c_2^* - c_3^* \geq 0$ and $- c_2^* \geq 0$ and $6 + 2c_2^* \geq 0$

i.e. if $c_2^* \geq \frac{2c_3^* - 24}{6}$ and $c_2^* \leq 0$ and $c_2^* \geq -3$

i.e. if $\max\left\{\frac{c_3^* - 24}{6}, -3\right\} \leq c_2^* \leq 0$ and c_3^* any real number.

When all the three components are changed together :

It $c_1 = 6$, $c = -2$, $c_3 = 3$ be changed respectively to c_1^*, c_2^*, c_3^*.

The modified table obtained from old optimal table is

			c_1^*	c_2^*	c_3^*	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
c_1^*	y_1	4	1	0	4	0	1
c_2^*	y_2	6	0	1	6	-1	2
			0	0	$4c_1^* - 6c_2^* - c_3^*$	$-c_2^*$	$c_1^* + 2c_2^*$

This table becomes an optimal table if all $z_j - c_j \geq 0$.

i.e. if $4c_1^* + 6c_2^* - c_3^* \geq 0$ and $- c_2^* \geq 0$ and $c_1^* + 2c_2^* \geq 0$

i.e. if $c_2^* \geq \frac{6c_3^* - 4c_1^*}{6}$ and $c_2^* \leq 0$ and $c_2^* \geq -\frac{c_1^*}{2}$

i.e. if $\max\left\{\frac{c_3^* - 4c_1^*}{6}, -\frac{c_1^*}{2}\right\} \leq c_3^* \leq 0$ and c_1^* any real number and c_3^* any real number.

4.4 Discrete Change In The Requirement Vector :

Let x_B be the optimal BFS of the LPP

Maximize $z = cx$

subject to $Ax = b, x \geq 0$

where $c, x^T \in R^n$, $b^T \in R^m$ and A is an max real matrix. We have $x_B = B^{-1}b$ and the net evaluations are the components $[c_B B^{-1} \ 1]\begin{bmatrix} A \\ -c \end{bmatrix}$. From these we see that x_B depends on b but net evaluations are independent of b: So change made in b will not affect optimality conditions i.e. optimal solution will remain optimal but it will change the solution x_B and it may become negative i.e. infeasible.

Let the component b_k of b be changed to $b_k^* = b_k + \Delta b_k$.

So the old solution $x_B = B^{-1}b$ becomes

$x_B^* = B^{-1} b^*$ where $b^* = [b_1, b_2, ..., b_{k-1}, b_k + \Delta b_k, b_{k+1}, ... , b_m]^T$

$$\text{Let} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & & b_{mm} \end{bmatrix}$$

$$\therefore x_B^* = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & & b_{mm} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{k-1} \\ b_k + \Delta b_k \\ b_{k+1} \\ \vdots \\ b_m \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \cdots & \cdots & & \cdots \\ \cdots & \cdots & & \cdots \\ \cdots & \cdots & & \cdots \\ b_{m1} & b_{m2} & & b_{mm} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{k-1} \\ b_k \\ b_{k+1} \\ \vdots \\ b_m \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2k} & \cdots & b_{2m} \\ \cdots & \cdots & & \cdots & \cdots & \\ \cdots & \cdots & & \cdots & \cdots & \\ \cdots & \cdots & & \cdots & \cdots & \\ b_{m1} & b_{m2} & & b_{mk} & \cdots & b_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= B^{-1} + \begin{bmatrix} b_{1k} & \Delta b_k \\ b_{2k} & \Delta b_k \\ \dots & \dots \\ b_{mk} & \Delta b_k \end{bmatrix}$$

$$\text{Thus } \begin{bmatrix} x^*_{B_1} \\ x^*_{B_2} \\ \vdots \\ x^*_{B_m} \end{bmatrix} = \begin{bmatrix} x_{B_1} \\ x_{B_2} \\ \vdots \\ x_{B_m} \end{bmatrix} + \begin{bmatrix} b_{1k} & \Delta b_k \\ b_{2k} & \Delta b_k \\ \vdots & \\ b_k & \Delta b_k \end{bmatrix}$$

$x^*_{B_i} = x_{B_i} + b_{ik}\Delta b_k$ for all i = 1, 2, ... , M

As we have noted this solution is optimal or better than optimal but may not be feasible shough basic.

Thus x^*_B will be an optimal BFS if $x^*_{B_i} \geq 0$ for all $i = 1, 2, ..., m$

i.e. if $x_{B_i} + b_{ik}\Delta b_k \geq 0$ for all $i = 1, 2, ... , m$

i.e. if $b_{ik}\Delta b_k \geq - x_{B_i}$ for all $i = 1, 2, ... , m$

For all $b_{ik} = 0$ this condition is satisfied.

For all $b_{ik} > 0$ this condition is satisfied if $\Delta b_k \geq -\frac{x_{B_i}}{b_{ik}}$

$\therefore$ We need $-\frac{x_{B_i}}{b_{ik}} \leq \Delta b_k$ for all $b_{ik} > 0$

i.e. we need max $\left\{-\frac{x_{B_i}}{b_{ik}} : b_{ik} > 0\right\} \leq \Delta b_k$

Again for all $b_{ik} < 0$ this condition is satisfied if $\Delta b_k \leq -\frac{x_{B_i}}{b_{ik}}$

$\therefore$ We need $\Delta b_k \leq -\frac{x_{B_i}}{b_{ik}}$ for all $b_{ik} < 0$

i.e. we need $\Delta b_k \leq \min\left\{-\frac{x_{B_i}}{b_{ik}} : b_{ik} < 0\right\}$

Hence x_{B_i} will be optimal basefeasible solution if Δb_k is selected satisfying the condition.

$$\max\left\{-\frac{x_{B_i}}{b_{ik}} : b_{ik} > 0\right\} \leq \Delta b_k \leq \min\left\{-\frac{x_{B_i}}{b_{ik}} : b_{ik} > 0\right\}$$

4.5 Illustrative Examples :

Example 4.5.1. Given the LPP

Maximize $z = -x_1 + 2x_2 - x_3$

Subject to $3x_1 + x_2 - x_3 \leq 10$

$-x_1 + 4x_2 + x_3 \geq 6$

$x_2 + x_3 \leq 4$

$x_1, x_2, x_3 \geq 0$

Determine the ranges for discrete changes of the components of b when changed one at a time, so as to maintain the optimality of the current optimum solution for the LPP.

Solution : Introducing slack variables $x_4 \geq 0$, $x_6 \geq 0$, surplus variable $x_5 \geq 0$ and artificial variable $x_7 \geq 0$ we have the standard form as follows

Maximize $z = -x_1 + 2x_2 - x_3 + ox_4 + ox_5 + ox_6 - Mx_7$

Subject to
$$3x_1 + x_2 - x_3 + x_4 = 10$$
$$-x_1 + 4x_2 + x_3 - x_5 + x_7 = 6$$
$$x_2 + x_3 + x_6 = 4$$
$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$$

The tables obtained by simplex method are as follows :

	c_j		-1	2	-1	0	0	0	$-M$
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
0	y_4	10	3	1	-1	1	0	0	0
$-M$	y_7	6	-1	$\boxed{4}$	1	0	-1	0	1
0	y_6	4	0	1	1	0	0	1	0
			$M+1$	$-4M-2$	$-M+2$	0	M	0	0
0	y_4	$\frac{17}{2}$	$\frac{13}{4}$	0	$-\frac{5}{4}$	1	$\frac{1}{4}$	0	$-\frac{1}{4}$
2	y_2	$\frac{3}{2}$	$-\frac{1}{4}$	1	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	$\frac{1}{4}$
0	y_6	$\frac{5}{2}$	$\frac{1}{4}$	0	$\frac{3}{4}$	0	$\boxed{\frac{1}{4}}$	1	$-\frac{1}{4}$
			$\frac{1}{2}$	0	$\frac{3}{2}$	0	$-\frac{1}{2}$	0	$M+\frac{1}{2}$
0	y_4	6	3	0	-2	1	0	-1	0
2	y_2	4	0	1	1	0	0	1	0
0	y_5	10	1	0	3	0	1	4	-1
			1	0	3	0	0	2	M

In this final table the basis in $B = [a_4\ a_2\ a_5]$ and in the initial table the basis is $I = [a_4\ a_7\ a_6]$

The inverse of the basis in the final table is given by

$$B^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

When b_1 is changed to $b_1 + \Delta b_1$ then the range $b\ \Delta b_1$ such that the optimality of the new BFS is not violated is given by

$$\max\left\{-\frac{x_{Bi}}{b_{i1}} : b_{ik} > 0\right\} \le \Delta b_1 \le \min\left\{-\frac{x_{Bi}}{b_{i1}} : b_{i1} < 0\right\}$$

$\therefore \max\left\{-\frac{x_{B1}}{b_{11}}\right\} \le \Delta b_1$ [$\because$ only $b_{11} = 1 > 0$ and there are x_0 negative b_{i1}]

or, $-\frac{6}{1} \le \Delta b_1$ or, $\Delta b_1 \ge -6$

$\therefore$ $b_1 + \Delta b_1 \ge b_1 - 6$

or, $b_1^* \ge 10 - 6$

or, $b_1^* \ge 4$

When b_2 is changed to $b_2 + \Delta b_2$ then the range of Δb_2 such that the optimality of the new BFS is not violated is given by

$$\max\left\{-\frac{x_{Bi}}{b_{i2}} : b_{i2} > 0\right\} \le \Delta b_2 \le \min\left\{-\frac{x_{Bi}}{b_{i2}} : b_{i2} > 0\right\}$$

$\therefore \Delta b_2 \le \min\left\{-\frac{x_{Bi}}{b_{i2}}\right\}$ [$\therefore$ only $b_{32} = -1 < 0$ and there is no positive b_{i2}]

or, $\Delta b_2 \le -\frac{10}{-1}$

or, $\Delta b_2 \le 10$

$\therefore$ $b_2 + \Delta b_2 \le b_2 + 10$

or, $b_2^* \le 6 = 10$

or, $b_2^* \le 16$

When b_3 is changed to $b_3 + \Delta b_3$ thin the ranges of Δb_3 such that the optimality of the new BFS is not violated are given by

$$\max\left\{-\frac{x_{Bi}}{b_{i3}} : b_{i3} > 0\right\} \le \Delta b_3 \le \min\left\{-\frac{x_{Bi}}{b_{i3}} : b_{i3} < 0\right\}$$

or, $\max\left\{-\frac{x_{B2}}{b_{23}}, \frac{x_{B3}}{b_{33}}\right\} \le \Delta b_3 \le \min\left\{-\frac{x_{B1}}{b_{13}}\right\}$

or, $\max\left\{-\frac{4}{1}, \frac{-10}{4}\right\} \le \Delta b_3 \le \min\left\{\frac{-6}{-1}\right\}$

or, $-\frac{5}{2} \le \Delta b_3 \le 6$

$\therefore\ -\frac{5}{2} + b_3 \le b_3 + \Delta b_3 \le 6 + b_3$

or, $-\frac{5}{2} + 4 \le b_3^* \le 6 + 4$

or, $\frac{3}{2} \le b_3^* \le 10$

Example 4.5.2 Consider the LPP

Maximize $z = 2x_1 + x_2 + 4x_3 - x_4$

subject to $x_1 + 2x_2 + x_3 - 3x_4 \le 8$

$x_2 + x_3 + 2x_4 \le 0$

$2x_1 + 7x_2 - 5x_3 - 10x_4 \le 21$

$x_1, x_2, x_3, x_4 \ge 0$

The optimal solution is it is contained in the following table

		c_j	2	1	4	−1	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
0	y_1	1	1	0	3	1	1	2	0
1	y_2	0	0	1	−1	−2	0	−1	0
0	y_7	5	0	0	−4	2	−2	3	1
		$z_j - c_j$	0	0	1	1	2	3	0

For each of the parameter change listed below, make the necessary correction in the optimal table and solve the resulting problem.

(a) change c_1 to 1

(b) change c to [1 2 3 4]

(c) change b to $[\ 3\ -2\ 4]^T$

(d) change b_2 to 11

(e) How much c_1 be changed without affecting the optimal solution.

Solution : (a) When c_1 is changed to 1 the modified form of the optimal table becomes

		c_j	1	1	4	-1	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
1	y_1	8	1	0	3	1	1	2	0
1	y_2	0	0	1	-1	-2	0	-1	0
0	y_7	5	0	0	-4	2	-2	3	1
		$z_j - c_j$	0	0	-2	0	1	1	0

From this table we see that changed solution is not optimal as $z_3 - c_3 < 0$. So we are to apply simplex method to get the optimal solution

		c_j	1	1	4	-1	0	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	min ratio
1	y_1	8	1	0	[3]	1	1	2	0	$\frac{8}{3}$
1	y_2	0	0	1	-1	-2	0	-1	0	
0	y_7	5	0	0	-4	2	-2	3	1	
			0	0	-2	0	1	1	0	
4	y_3	$\frac{8}{3}$	$\frac{1}{3}$	0	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	0	
1	y_2	$\frac{8}{3}$	$\frac{1}{3}$	1	0	$-\frac{5}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	0	
0	y_7	$\frac{95}{3}$	$\frac{4}{3}$	0	0	$\frac{10}{3}$	$-\frac{2}{3}$	$\frac{17}{3}$	1	
$z =$	$\frac{40}{3}$	$z_j - c_j$	$\frac{2}{3}$	0	0	$\frac{2}{3}$	$\frac{5}{3}$	$\frac{7}{3}$	0	

Since all $z_j - c_j \geq 0$, this optimality conditions are satisfied. The optimal solution is $x_1 = 0$, $x = \frac{8}{3}$, $x_3 = \frac{8}{3}$, $z_{max} = \frac{40}{3}$.

When c is changed from [2 1 4 – 1] to [1 2 3 4] this modified form of the optimal table becomes

		c_j	1	2	3	4	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
1	y_1	8	1	0	3	1	1	2	0
2	y_2	0	0	1	− 1	− 2	0	− 1	0
0	y_7	5	0	0	− 4	2	− 2	3	1
		$z_j - c_j$	0	0	− 2	− 7	1	0	0

We see that there are negative $z_j - c_j$ viz $z_3 - c_3 = -2$ and $z_4 - c_4 = -7$. Hence the solution is not optimal. We apply simplex method to get the optimal solution.

		c_j	1	2	3	4	0	0	0	Min ration
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	
1	y_1	8	1	0	3	1	1	2	0	8
2	y_2	0	0	1	− 1	− 2	0	− 1	0	–
0	y_7	5	0	0	− 4	[2]	− 2	3	1	$\frac{5}{2}$
		$z_j - c_j$	0	0	− 2	− 7	1	0	0	
1	y_1	$\frac{11}{2}$	1	0	[5]	0	2	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{11}{10}$
2	y_2	5	0	1	− 5	0	− 2	2	1	–
4	y_7	$\frac{5}{2}$	0	0	− 2	1	− 1	$\frac{3}{2}$	$\frac{1}{2}$	–
		$z_j - c_j$	0	0	− 16	0	− 6	$\frac{21}{2}$	$\frac{7}{2}$	
3	y_3	$\frac{11}{10}$	$\frac{1}{5}$	0	1	0	$\frac{2}{5}$	$\frac{1}{10}$	$-\frac{1}{10}$	
2	y_2	$\frac{21}{2}$	1	1	0	0	0	$\frac{5}{2}$	$\frac{1}{2}$	
4	y_7	$\frac{47}{10}$	$\frac{2}{5}$	0	0	1	$\frac{1}{5}$	$\frac{17}{10}$	$\frac{3}{10}$	
$z = \frac{431}{10}$		$z_j - c_j$	$\frac{16}{5}$	0	0	0	$\frac{2}{5}$	$\frac{121}{10}$	$\frac{19}{10}$	

Since all $z_j - c_j \geq$ we have obtained this optimal table. The optimal solution is

$$x_1 = 0,$$

$$x_2 = \frac{21}{12},$$

$$x_3 = \frac{11}{10},$$

$$x_4 = \frac{47}{10}$$

and $z_{max} = \frac{431}{10} = 43\frac{1}{10}$.

..... initial table the basis is $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_5 & a_6 & a_7 \end{bmatrix}$ and so inverse of the basis of the final table is given by $B^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix}$.

The new solution when b is changed from $[8\ 0\ 21]^T$ to $[3\ -2\ 4]^T$ is given by

$$x_B^* = B^{-1} b^* = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ -8 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_7 \end{bmatrix}$$

This solution is not feasible but optimal. Hence to get the optimal solution we are to apply dual simplex method. The following are the modified optimal table and tables obtained by dual simplex method.

		c_j							
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
2	y_1	−1	1	0	3	1	1	2	0
1	y_2	2	0	1	−1	−2	0	−1	0
0	y_7	−8	0	0	$\boxed{-4}$	2	−2	3	1
		$z_j - c_j$	0	0	1	1	2	3	0
$\frac{z_j - c_j}{y_{3j}} : y_{3j} < 0$				$-\frac{1}{4}$		−1			
2	y_1	−7	1	0	0	$\frac{5}{2}$	$\boxed{-\frac{1}{2}}$	$\frac{17}{4}$	$\frac{3}{4}$
1	y_2	4	0	1	0	$-\frac{5}{2}$	$\frac{1}{2}$	$-\frac{7}{4}$	$-\frac{1}{4}$
4	y_3	2	0	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{1}{4}$
		$z_j - c_j$	0	0	1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{15}{4}$	$\frac{1}{4}$
$\frac{z_j - c_j}{y_{3j}} : y_{ij} < 0$				0		−3			
0	y_5	−14	−2	0	0	−5	1	$\frac{17}{4}$	$-\frac{3}{2}$
1	y_2	−3	1	1	0	0	0	$\frac{5}{2}$	$-\frac{1}{2}$
4	y_3	−5	1	0	1	2	0	$\frac{7}{2}$	$\frac{1}{2a}$
		$z_j - c_j$	3	0	1	9	0	$\frac{33}{2}$	$\frac{5}{2}$

We note here that $x_{B_3} = -5 < 0$ but all $y_{3j} \geq 0$. Hence this changed problem has no feasible solution.

When b_2 changed to 11, the new solution is given by

$$x_B^* = \begin{bmatrix} x_1 \\ x_2 \\ x_7 \end{bmatrix} = B^{-1} b^* = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 11 \\ 21 \end{bmatrix} = \begin{bmatrix} 30 \\ -11 \\ 38 \end{bmatrix}$$

Since $x_{B_2}=-11<0$, the solution is not feasible but optimal. So to get optimal solution we are to apply dual simplex method in the modified optimal table. The dual simplex tables are as follows :

	c_j		2	1	4	−1	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
2	y_1	30	1	0	3	1	1	2	0
1	y_2	−11	0	1	−1	−2	0	−1	0
0	y_7	38	0	0	−4	2	−2	3	1
		z_j-c_j	0	0	1	1	2	3	0
$\frac{z_j-c_j}{y_{2j}}: y_{2j}<0$					−1	$-\frac{1}{2}$		3	
2	y_1	$\frac{49}{2}$	1	$\frac{1}{2}$	$\frac{5}{2}$	0	1	$\frac{3}{2}$	0
−1	y_4	$\frac{11}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0
0	y_7	27	0	1	−5	0	−2	2	1
$z=\frac{87}{2}$		z_j-c_j	0	$\frac{1}{2}$	$\frac{1}{2}$	0	2	$\frac{5}{2}$	0

In this tabl all $x_{B_i}>0$ and all $z_j-c_j\geq 0$. So we have reached to the optimal table.

The optimal solution is $x_1=\frac{49}{2}, x_2=0, x_3=0, x_4=\frac{11}{2}$ and $z_{max}=\frac{87}{2}$.

(e) When $c_1=1$ is replaced by c_1^* the modified form of the optimal table is given by

	c_1^*		1	4	4	−1	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
c_1^*	y_1	8	1	0	3	1	1	2	0
1	y_2	0	0	1	−1	−2	0	−1	0
0	y_7	5	0	0	−4	2	−2	3	1
		z_j-c_j	0	0	$3c_1^*-5$	c_1^*-1	c_1^*	$2c_1^*-1$	0

This table remains as optimal tablie if all $z_j - c_j \geq 0$

i.e. if $3c_1^* - 5 \geq$ and $c_1^* - 1 \geq 0$ and $c_1^* \geq 0$ and $2c_1^* - 1 \geq 0$

i.e. if $c_1^* \geq \frac{5}{3}$ and $c_1^* \geq$ and c_1^* and $c_1^* \geq \frac{1}{2}$

i.e. if $c_1^* \geq \frac{5}{3}$.

(e) Alternative method using formula :

Since $c_1 \in c_B$, the range of Δc_1 for which the optimality of the solution is maintained is given by

$$\max\left\{\frac{z_j - c_j}{y_{1j}} : y_{1j} > 0\right\} \leq \Delta c_1 \leq \min\left\{\frac{z_j - c_j}{y_{1j}} : y_{1j} < 0\right\}$$

$$i.e.\ \max\left\{-\frac{z_3 - c_3}{y_{13}}, -\frac{z_4 - c_4}{y_{14}}, -\frac{z_5 - c_5}{y_{15}}, -\frac{z_5 - c_5}{y_{16}}\right\} \leq \Delta c_1$$

$$i.e.\ \max\left\{-\frac{1}{3}, -\frac{1}{1}, -\frac{2}{1}, -\frac{3}{2}\right\} \leq \Delta c_1$$

$$i.e.\ -\frac{1}{3} \leq \Delta c_1 < \infty$$

$$\therefore\ c_1 - \frac{1}{3} \leq c_1 + \Delta c_1 < c_1 + \infty$$

$$\text{or, } 2 - \frac{1}{3} \leq c_1^* < \infty$$

$$\text{or, } \frac{5}{3} \leq c_1^* < \infty$$

$\therefore$ If $c_1^* \geq \frac{5}{3}$ the optimal solution remain optimal.mα

4.6 Addition Of A Single Variable :

Let the optimal solution of the given LPP

Maximize $z = cx$

subject to $Ax = b, x \geq 0$

be known. Let x_{n+1} be added with it and the coefficient vector associated with x_{n+1} be a_{n+1} and the cost coefficient for x_{n+1} be c_{n+1}.

Since b is not changed the old optimal solution will be feasible solution of the new LPP but it may not be optimal. Let B be the optimal basis and C_B be the associated cost vector of the old LPP. Then they are also the same for the new LPP. It is optimum for the new LPP if $z_{n+1} - C_{n+1} \geq 0$.

In case $z_{n+1} - C_{n+1} < 0$, x_{n+1} will enter the solution and simplex method is to be applied to the old optimal table added with $(n + 1)$th column as $y_{n+1} = B^{-1} a_{n+1}$.

4.7 Illustrative Example :

Example 4.7.1: Consider the LPP

Maximize $z = x_1 + 2x_2 + x_3$

subject to $2x_1 + x_2 - x_3 \leq 2$

$2x_1 - x_2 + 5x_3 \leq 6$

$4x_1 + x_2 + x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

Let a new variable $x_3' \geq 0$ be introduced with cost (i) 3 (ii) 5 and $a_3' = [2 - 1 - 4]$. Discuss the effect.

The solution of the LPP is obtained by simplex method. The following are the tables.

		c_j	1	2	1	−1	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	min ratio
0	y_4	2	2	[1]	−1	1	0	0	2
0	y_5	6	2	−1	5	0	1	0	–
0	y_6	6	4	1	1	0	0	1	–
		$z_j - c_j$	−1	−2	−1	0	0	0	
2	y_2	2	2	−1	1	0	0	0	
0	y_5	8	4	0	[4]	1	1	0	2
0	y_6	4	2	2	−1	0	0	1	2
		$z_j - c_j$	3	−3	2	0	0	0	
2	y_2	4	3	1	0	$\frac{5}{4}$	$\frac{1}{4}$	0	
1	y_3	2	1	0	1	$\frac{1}{4}$	$\frac{1}{4}$	0	
2	y_6	0	0	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	1	
$z = 10$		$z_j - c_j$	6	0	0	$\frac{1}{4}$	$\frac{3}{4}$	0	

$\therefore$ The optimal solution is $x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$

The inverse of the basis in the optimal table is

$$B^{-1} = \begin{bmatrix} \frac{5}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{3}{2} & -\frac{1}{2} & 2 \end{bmatrix}$$

The added column for x_3' is $a_3' = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

The corresponding column in the final table is given by

$$y_3' = B^{-1} a_3' = \begin{bmatrix} \frac{5}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{3}{2} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{9}{4} \\ \frac{1}{4} \\ \frac{6}{4} \end{bmatrix}$$

(i) When $c_3' = 3$, we have

$$z_3' - c_3' - c_B y_3' - c_3' = [2\ 1\ 0] \begin{bmatrix} \frac{9}{4} \\ \frac{1}{4} \\ \frac{6}{4} \end{bmatrix} - 3 = \frac{19}{4} - 3 = \frac{7}{4} > 0$$

∴ The optimatity condition is satisfied for the changed problem also. The optimal solution is $x_1 = 0$, $x_2 = 4$, $x_3 = 2$.

(ii) When $z_3' - c_3' = c_B y_3' - c_3' = [2\ 1\ 0] \begin{bmatrix} \frac{9}{4} \\ \frac{1}{4} \\ \frac{6}{4} \end{bmatrix} = \frac{19}{4} - 5 = -\frac{1}{4} < 0$

∴ Optimatity condition is not satisfied here.

We shall modify the optimal table of the old problem with added column $y_3' = \begin{bmatrix} \frac{9}{4} \\ \frac{1}{4} \\ \frac{6}{4} \end{bmatrix}$

and $z'_3 - c'_3 = -\frac{1}{4}$ and $c'_3 = 5$. Then to get the optimal solution we are to apply simplex method. The tables obtained are as follows.

		c_j	1	2	1	5	0	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y'_3	y_4	y_5	y_6	min ratio
2	y_2	4	3	1	0	$\frac{9}{4}$	$\frac{5}{4}$	$\frac{1}{4}$	0	$\frac{16}{9}$
1	y_3	2	1	0	1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	8
0	y_6	0	0	0	0	$\boxed{\frac{6}{4}}$	$-\frac{3}{2}$	$-\frac{1}{2}$	1	0 →
		$z_j - c_j$	6	0	0	$-\frac{1}{4}$	$\frac{11}{4}$	$\frac{3}{4}$	0	
2	y_2	4	3	1	0	0	$\frac{7}{2}$	1	$-\frac{3}{2}$	
1	y_3	2	1	0	1	0	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{6}$	
5	y'_3	0	0	0	0	1	-1	$-\frac{1}{3}$	$\frac{2}{3}$	
$z = 10$		$z_j - c_j$	6	0	0	0	$\frac{5}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	

In this table all $x_{B_i} \geq 0$ and all $z_j - c_j \geq$. So the optimatily conditions are satisfied. The optimal solution is given by $x_1 = 0$, $x_2 = 4$, $x_3 - 2$, $x'_3 = 0$ and $z_{max} = 10$.

4.8 Deletion of A Variable :

From a LPP if we delete a variable them two cases any arise.

Case 1. If this variable deleted is non basic then the feasibility and optimality conditions are not affected. So the optimal solution of the old problem is the optimal solution of the new problem.

Case 2. If the variable deleted is basic then the conditions of optimality may be affected and so a new solution is to be obtained. For this new optimal solution, we are assign a cost – M coresopnding to the basic variable to be deleted and apply simplex method after modifying the old optimal table.

4.9 Illustrative Example :

Example 4.9.1 For the LPP

Maximize $z = x_1 + 2x_2 + x_3$

subject to $2x_1 + x_2 - x_3 \leq 2$

$2x_1 - x_2 + 5x_3 \leq 6$

$4x_1 + x_2 + x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

the optimal table is

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
2	y_2	4	3	1	0	$\frac{5}{4}$	$\frac{1}{4}$	0
1	y_3	2	1	0	1	$\frac{1}{4}$	$\frac{1}{4}$	0
0	y_6	0	0	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	1
$z = 10$		$z_j - c_j$	6	0	0	$\frac{1}{4}$	$\frac{3}{4}$	0

Discuss the effect of deletion of the variable (i) x_1 (ii) x_2 (iii) x_3.

(i) From the optimal table we see that x_1 is a deleted the optimal solution remains unaffected. Hence old optimal solution is also the new optimal solution is

$x_1 = 0, x_2 = 4, x_3 = 2$ & $z_{max} = 10$

(ii) From the optimal table we see that x_2 is a basic variable. Hence we make a new starting table by changing $c_2 = 2$ by $-$ M, where M is a big positive number. As M is very large the optimatity conditions are not affected and once it goes out from the basis it never reappears in the basis in the simplex method.

The modified table is

		c_j	1	$-M$	1	0	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	min ratio
$-M$	y_2	4	3	1	0	$\frac{5}{4}$	$\frac{1}{4}$	0	$\frac{4}{3}$
1	y_3	2	1	0	1	$\frac{1}{4}$	$\frac{1}{4}$	0	2
0	y_6	0	0	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	1	...
			$-3M$	0	0	$-\frac{5M}{4}+\frac{1}{4}$	$-\frac{M}{4}+\frac{1}{4}$	0	
1	y_1	$\frac{4}{3}$	1	$\frac{1}{3}$	0	$\frac{5}{12}$	$\frac{1}{12}$	0	
1	y_3	$\frac{2}{3}$	0	$-\frac{1}{3}$	1	$-\frac{1}{6}$	$\frac{1}{6}$	0	
0	y_6	0	0	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	1	
$z = 2$		$z_j - c_j$	0	M	0	$\frac{1}{4}$	$\frac{1}{6}$	0	

In this table all $z_j - c_j \geq$ and all $x_{B_i} \geq 0$, so we have reached to optimal table. The optimal solution is

$$x_1 = \frac{4}{3}, x_2 = 0, x_3 = \frac{2}{3}$$

and $z_{max} = 2$.

(iii) From the optimal table we see that x_3 is a basic variable. Hence we make a new starting table by changing $c_1 = 1$ by $-M$, where M is a big positive number. As M is very large the optimality conditions are satisfied. Also once it goes out from the basis it never reappears in the basis in the simplex method. The modified table and other simplex tables are as follows :

		c_j	1	$-M$	1	0	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	min ratio
2	y_2	4	3	1	0	$\frac{5}{4}$	$\frac{1}{4}$	0	$\frac{4}{3}$
$-M$	y_3	2	1	0	1	$\frac{1}{4}$	$\frac{1}{4}$	0	2
0	y_6	0	0	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	1	...
			$-M+5$	0	0	$-\frac{5M}{4}+\frac{5}{2}$	$-\frac{M}{4}+\frac{1}{2}$	0	
1	y_1	$\frac{4}{3}$	1	$\frac{1}{3}$	0	$\frac{5}{12}$	$\frac{1}{12}$	0	16
$-M$	y_3	$\frac{2}{3}$	0	$-\frac{1}{3}$	1	$-\frac{1}{6}$	$\frac{1}{6}$	0	4
0	y_6	0	0	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	1	...
1	y_1	1	1	$\frac{1}{2}$		$\frac{1}{2}$	0	0	2.
0	y_5	4	0	-2		-11	1	0	...
0	y_6	2	0	-1		-2	0	1	...
2	y_2	2	2	1		1	0	0	
0	y_5	8	2	0		$-\frac{3}{2}$	1	0	
0	y_6	4	0	0		$-\frac{3}{2}$	0	1	
		4	3	0		2	0	0	

The optimal table is obtained and this optimal solution is $x_1 = 0$, $x_2, = 2$, $x_3 = 0$ and $z_{max} = 4.\alpha$

4.10 Addition Of A New Constraint :

Addition of a new constraint may or may not affect the current optimal solution. Two cases will arise.

(i) If the added constraint is satisfied by the old optimal solution then teh old optimal solution is also the new optimal solution.

(ii) If the added constraiant is not satisfied by the old optimal solution, then this old optimal solution becomes an infeasible solution for the new problem.

To obtain the optimal solution for the changed problem we are first to modify the fimal table and then apply dual simplex method.

The following three situations will arise depending on the nature of the solution to the original LPP.

If original LPP has an optimal solution then the modified LPP may have an optimal solution or it will give no F.S.

If the original LPP has unbounded solution then the modified LPP may have optimal solution or it will have no F.S. or it will have unbounded solution.

If the original LPP has no F.S. then the modified LPP will have also no F.S.α

4.11 Illustrative Examples :

Example 4.11.1 Let us consider the final table of a LPP

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
2	y_1	3	1	0	0	– 1	0	·5	·2	– 1
4	y_2	1	0	1	0	2	1	– 1	0	·5
1	y_3	7	0	0	1	1	– 2	5	– ·3	2
		$z_j - c_j$	0	0	0	– 1	0	2	1	2

where y_6, y_7 and y_8 are slack variables.

If the constraint

(i) $2x^1 + 3x^2 - x^3 + 2x^4 + 4x^5 \le 5$

(ii) $2x^1 + 3x^2 - x^3 + 2x^4 + 4x^5 \le 1$

is added then find the solution of the changed LPP.

Solution :

From the final table we see that the optimal solution of the old LPP is

$$x_1 = 3, x_2 = 1, x_3 = 7, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0, x_8 = 0$$

(i) The added constraint is

$$2x_1 + 3x_2 - x_3 + 2x_4 + 4x_5 \le 5$$

Putting $x_1 = 3, x_2 = 1, x_3 = 7, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0, x_8 = 0$ in this constraint we have

$$2.3 + 3.1 - 7 + 2.0 + 4.0 \le 5$$

or, $6 + 3 - 7 \le 5$

or, $2 \le 5$.

This is true. So the solution satisfies the added constraint. Hence the old optimal solution is also optimal solution to the new problem.

The added constraint is

$$2x_1 + 3x_2 - x_3 + 2x_4 + 4x_5 \le 1$$

Putting $x_1 = 3, x_2 = 1, x_3 = 7, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0, x_8 = 0$ in this constraint we get

$$2.3 + 3.1 - 7 + 2.0 + 4.0 \le 1$$

or, $6 + 3 - 2 \le 1$

or, $2 \le 1$

This is not time *i.e.* the optimal solution to the old problem does not satisfy the added constraint. To get the solution of the new LPP we introduces the new constraint with a new slack variable in the optimal table of the old problem. We then modify this table to have a unit basis and then apply dual simplex method to it. The following are the tables.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9
2	y_1	3	1	0	0	− 1	0	·5	·2	− 1	0
4	y_2	1	0	1	0	2	1	− 1	0	·5	0
1	y_3	7	0	0	1	− 1	− 2	5	− ·3	2	0
0	y_9	1	2	3	− 1	2	4	0	0	0	1
		$z_j - c_j$	0	0	0	2	0	2	·1	2	0
2	y_1	3	1	0	0	− 1	0	·5	·2	− 1	0
4	y_2	1	0	1	0	2	1	− 1	0	·5	0
1	y_3	7	0	0	1	− 1	− 2	5	− ·3	2	0
0	y_9	− 1	0	0	0	− 3	[− 1]	7	− ·7	2·5	1
			0	0	0	2	0	2	·1	2	0
$\frac{z_j - c_j}{y_{4j}} : y_{4j} < 0$						$-\frac{2}{3}$	0		$-\frac{\cdot 1}{7}$		
2	y_1	3	1	0	0	− 1	0	·5	·2	− 1	0
4	y_2	0	0	1	0	− 1	0	6	− ·7	3	0
1	y_3	9	0	0	1	5	0	− 9	1·1	− 3	− 2
2	y_5	1	0	0	0	3	1	− 7	·7	− 2·5	− 1
			0	0	0	2	0	2	·1	2	0

The second table is obtained by the operation $R_4^1 = R_4 - 2R_1 - 3R_2 + R_3$. The third table is obtained by using dual simplex method to the second table and is the the final table. The optimal solution is $x_1 = 3, x_2 = 0, x_3 = 9, x_4 = 0, x_5 = 1$.

4.12 Summary :

The usefulness of post-optimality analysis is discussed. Then onle by one the different situations viz discrete changes in the cost vector and requirement vector, addition and deletion of a single variable, and addition of a new constraint are discussed. Each situation is illustrated by examples.

4.13 Self Assessment Questions :

1. For the LPP

$$\begin{aligned} \text{Maximize} \quad & z = 15x_1 + 45x_2 \\ \text{subject to} \quad & 5x_1 + 2x_2 \le 162 \\ & x_1 + 16x_2 \le 240 \\ & x_2 \le 50 \\ & x_1, x_2 \ge 0 \end{aligned}$$

find the optimal solution. Find the range of each cost coefficient (changed one at a time) to give same optimal solution.

[Ans : x_1 =352/13, x_2 = 173/13, z_{max} = 1005]

2. Find how much the 7 in the first constraint of the problem

$$\begin{aligned} \text{Minimize} \quad & z = x_1 - 3x_2 + 2x_3 \\ \text{subject to} \quad & 3x_1 - x_2 + 2x_3 \le 7 \\ & -2x_1 + 4x_2 \le 12 \\ & -4x_1 + 3x_2 + 8x_3 \le 10 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

be changed before the basis of the optimal table would change.

3. Find the optimal solution of the LPP and the separate ranges of variations of b_2 and b_3 consistent with the optimatity of the solution

$$\begin{aligned} \text{Minimize} \quad & z = -x_1 + 2x_2 - x_3 \\ \text{subject to} \quad & 3x_1 + x_2 - x_3 \le 10 \\ & -x_1 + 4x_2 + x_3 \ge 6 \\ & x_2 + x_3 \le 4 \\ & x_1, x_2, x_3 \ge 0. \end{aligned}$$

Determine also this efficient discrete changes in the components of the cost vector which correspond to the basic variables.

[Ans : $x_1 = 0$, $x_2 = 4$, $x_3 = 0$; $\Delta b_2 \le 10$, $-5/2 \le \Delta b_3 \le 6$, $-2 \le \Delta c_2$]

4. Following is the optimal table for an LPP

		c_j	2	1	1	2	0
c_B	B	x_B	y_1	y_2	y_3	y_4	y_5
2	a_1	3	1	0	– 1	3	2
1	a_2	4	0	1	4	– 1	– 2
			0	0	1	3	2

(i) Find the limitations of this values of c_3, c_4, c_5 (taking one at a time) for which the current solution will remain optimal.

(ii) Find the optimal solution to the problem, if c_3 is changed to 3.

(iii) Find the limitations of the values of c_1 for which the current solution remains optimal.

(iv) Find the optimal solution to this problem, if c_1 is changed to 5.

[Ans : (i) $-\alpha < c_3 \le 2, -\alpha < c_4 \le 5, -\alpha < c_5 \le 2$

(ii) $x_1 = 4, x_3 = 1, x_2 = 0, x_4 = 0$

(iii) $1 \le c_1 \le 3$

(iv) $x_1 = 13/4, x_2 = 0, x_3 = 1, x_4 = 0$]

5. Find the optimal solution of the IPP

Maximize $z = 4x_1 + 3x_2$

subject to $x_1 + x_2 \le 5$

$3x_1 + x_2 \le 7$

$x_1 + 2x_2 \le 10$

$x_1, x_2 \ge 0$

Show how to find the optimal solution of the problem, if

(i) the first component of the original requirement vector be increased by one unit and the third component be decreased by one unit.

(ii) the second component of the original requirement vector be decreased by two units.

[Ans : (i) $x_1 = 1, x_2 = 4, z_{max} = 16$

(ii) $x_1 = 0, x_2 = 5, z_{max} = 15$]

Unit 5 ❑ Quadratic Programming Problem

Structure

5.1 Introduction :

Quadratic programiing problem is the most well behaved nonlinear programming problem. Quadratic programming deals with non-linear programming problem of maximizing (or minimizing) quadratic objective function subject to a set of linear inequality constraints. The solution of this problem is based on the Kuhn-Tucker conditions. The quadratic objective function to be optimized is taken as strictly convex for minimization and strictly concave for maximization. As the solution space is always convex, the optimal the solution obtained is global is nature.

Definition 5.1.1 : Let x^T and $C \in R^n$ and Q be a symmetric $n \times n$ real matrix then, the problem quadratic pogamming problem is

$$\text{Maximize (or minimize) } f(x) = cx + \frac{1}{2}x^T Q x$$

$$\text{subject to } Ax \leq b$$

$$x \geq 0$$

where $x = [x_1, x_2, ..., x_n]^T$

$c = [c_1, c_2, ..., c_n]$

$b = [b_1, b_2, ... b_m]^T$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ and } Q = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

The function $x^T Qx$ defines a quadratic form when Q is a symmetric matrix.

The quadratic form $x^T Qx$ is said to be positive-definite if $x^T Qx \geq$ for all $x \neq 0$.

The quadratic form $x^T Qx$ is said to be positive semi definite if $x^T Qx \geq$ for at one $x \neq 0$.

The quadratic form $x^T Qx$ is said to be negative definite and negative semi-definite if $-x^T Qx$ is positive definite and positive semi-definite respectively.

In quadratic programming problem $x^T Qx$ is assmed to be negative definite in the maximization case, and positive definite in the minimization case. These means that $f(x) = cx + \frac{1}{2}x^T Qx$ is assumed to be strictly convex function for minimmzation case and strictly concave fomaximization case.

As the constraints are always assumed to be linear, the solution space of a quadratic programming problem is always convex.

Thus the solution obtained using Kuhn-Tucker conditions given global optimum of the quadatic programming problem.

5.2 Kuhn-Tucker Conditions for Quadratic Programming Problem :

Let the quadratic programming problem be

$$\text{Maximize} \quad f(x) = \sum_{j=1}^{n} c_j x_j \frac{1}{2} \sum_{j=1}^{n} \sum_{j=1}^{n} c_{jk} x_j x_k$$

subject to the constraints

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = 1, 2, \ldots\ldots, m$$

and $x_j \geq 0, j = 1, 2, \ldots\ldots, n$

where $c_{jk} = c_{kj}$ for all j and k.

Introducing slack variables q_i^2 and r_j^2 the problem reduces to

$$\text{Maximize } f = \sum_{j=1}^{n} c_j x_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} c_{jk} x_j x_k$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij} x_j - b_i + q_i^2 = 0, \; i = 1, 2, \ldots\ldots m$$

$$-x_j + r_j^2 = 0, \; j = 1, 2, \ldots\ldots n.$$

The Lagangian function is given by

$$L(x_1, x_2, \ldots, x_n, q_1, q_2, \ldots, q_m, r_1, r_2, \ldots, r_n, \lambda_1, \lambda_2, \ldots, \lambda_m, \mu_1, \mu_2, \ldots, \mu_n)$$

$$= \sum_{j=1}^{n} c_j x_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} c_{jk} x_j x_k - \sum_{i=1}^{m} \lambda_j \left(\sum_{j=1}^{n} a_{ij} x_j - b_i + q_i^2 \right) - \sum_{j=1}^{n} \mu_j \left(-x_j + r_j^2 \right)$$

The Kuhn-Tucher conditions are given by

$$\frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i a_{ij} - \mu_j(-1) = 0, \quad j = 1, 2, \ldots\ldots, n$$

$$\lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right) = 0, \quad i = 1, 2, \ldots\ldots, m$$

$$\mu_j x_j = 0, \quad j = 1, 2, \ldots\ldots, n$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_j, \quad i = 1, 2, \ldots\ldots, m$$

$$x_j \geq 0, \quad j = 1, 2, \ldots\ldots, m$$

$$\lambda_i \geq 0, \quad i = 1, 2, \ldots\ldots, m$$

$$\mu_j \geq 0, \quad j = 1, 2, \ldots\ldots, n$$

Letting $q_i^2 = s_i \geq 0$ these equations becomes

$$\left.\begin{array}{l} c_j + \sum_{k=1}^{n} c_{jk}\, x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j = 0 \\ \sum_{j=1}^{n} a_{ij}\, x_j - b_i + s_i = 0, i = 1,2,\ldots,m \end{array}\right\} \quad \ldots \quad \ldots \quad (1)$$

$$\left.\begin{array}{l} \lambda_i s_i = 0, i = 1,2,\ldots,m \\ \mu_j x_j = 0, j = 1,2,\ldots,n \end{array}\right\} \quad \ldots \quad \ldots \quad (2)$$

$$\left.\begin{array}{l} \lambda_i \geq 0, i = 1,2,\ldots,m \\ \mu_j \geq 0, j = 1,2,\ldots,n \\ x_j \geq 0, j = 1,2,\ldots,n \\ s_i \geq 0, i = 1,2,\ldots,n \end{array}\right\} \quad \ldots \quad \ldots \quad (3$$

(1) is a system of $m + n$ linear equations in x_j, λ_i, μ_j and s_i.

The solution of these system which will satisfy also (2) and (3) is the required optimal solution of the quadrative programming problem.

5.3 Wolfe's Modified Simplex Method :

To solve the system (1) satisfying the conditions (2) and (3) Wolfe suggested to introduce the non-negative artificial variables $\beta_1, \beta_2, \ldots\ldots \beta_n$ in the Kuhn-Tucker conditions (1) and to constract an objective function $z = -\beta_1 - \beta_2 - \ldots - \beta_n$ and to consider the following LPP with complementary slackness condition.

Maximize $\quad z = -\beta_1 - \beta_2 - \ldots\ldots - \beta_n$

subject to $\quad \sum_{k=1}^{n} c_{jk} x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j = -c_j, j = 1, c, \ldots, n$

$$\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i, i = 1,2,\ldots\ldots,m$$

$$\lambda_1, s_i, x_j, \mu_i, \mu_j, \beta_j \geq 0, i = 1,2,\ldots\ldots,m, j = 1,2,\ldots\ldots,n$$

and satisfying the complementarhy slackness conditions

$$\lambda_i s_i = 0, i = 1,2,\ldots\ldots,m$$
$$\mu_j x_j = 0, j = 1,2,\ldots\ldots,n$$

The optimum solution of theis LPP gives the optimum solution of the given QPP.

Note : To maintain the condition $\lambda_i s_i = 0 = \mu_j x_j$ all the time we should note that if λ_i is in the basic solution with positive value then s_i can not be basic with positive value. Similarly μ_j and x_j cannot be in the basic solution (i.e. positive) simultaneously.

Example 5.3.1 Using Wolfe's method solve the quadratic programming problem

$$\text{Maximize } z = 2x_1 + x_2 - x_1^2$$
$$\text{subject to} \quad 2x_1 + 3x_2 \le 6$$
$$2x_1 + x_2 \le 4$$
$$x_1, x_2 \ge 0$$

Solution : First be write all constraints with '≥' sign to get the problem as

$$\text{Maximize } z = 2x_1 + x_2 - x_1^2$$
$$\text{subject to} \quad 2x_1 + 3x_2 \le 0$$
$$2x_1 + x_2 \le 4$$
$$x_1 \le 0$$
$$-x_2 \le 0$$

Introducing slack variable q_1^2, q_2^2, r_1^2 and r_2^2 we get

$$\text{Maximize } z = 2x_1 + x_2 - x_1^2$$
$$\text{subject to} \quad 2x_1 + 3x_2 + q_1^2 = 6$$
$$2x_1 + x_2 + q_2^2 = 4$$
$$-x_1 + r_1^2 = 0$$
$$-x_2 + r_2^2 = 0$$

We now constuct the Largrange function

$$L\ (x_1, x_2, q_1, q_2, r_1, r_2, \lambda_1, \lambda_2, \mu_1, \mu_2)$$
$$= \left(2x_1 + x_2 - x_1^2\right) - \lambda_1\left(2x_1 + 3x_2 + q_1^2 - 6\right) - \lambda_2\left(2x_1 + x_2 + q_2^2 - 4\right)$$
$$-\mu_1\left(-x_1 + r_1^2\right) - \mu_2\left(-x_2 + r_2^2\right)$$

The Kuhn-Tucker's necessay and sufficient conditions gives

$$\frac{\partial L}{\partial x_1} = 0 \text{ or, } 2 - 2x_1 - 2\lambda_1 - 2\lambda_2 + \mu_1 = 0$$

$$\frac{\partial L}{\partial x_2} = 0 \text{ or, } 1 - 3\lambda_1 - \lambda_2 + \mu_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = 0 \text{ or, } 2x_1 3x_2 + q_1^2 - 6 = 0$$

$$\frac{\partial L}{\partial \lambda_2} = 0 \text{ or, } 2x_1 + x_2 + q_2^2 - 4 = 0$$

$$\lambda_1 q_1^2 - 0,\ \lambda_2 q_2^2 = 0,\ \mu_1 x_1 = 0,\ \mu_2 x_2 = 0$$

$$x_1, x_2, \lambda_1, \lambda_2, \mu_1, \mu_2 \geq 0$$

Taking $q_1^2 = s_1$ and $q_2^2 = s_2$ we get

$$\begin{aligned}
2x_1 + 2\lambda_1 - \mu_1 &= 2 \\
3\lambda_1 + \lambda_2 - \mu_2 &= 1 \\
2x_1 + 3x_2 \quad + s_i &= 6 \\
2x_1 + x_2 \quad + s_2 &= 6
\end{aligned}$$

$$\lambda_1 s_1 = 0,\ \lambda_2 s_2 = 0,\ \mu_1 x_1 = 0,\ \mu_2 x_2 = 0$$

$$x_1, x_2, \lambda_1, \lambda_2, \mu_1, \mu_2, s_1, s_2 \geq 0$$

With necessary modification we use phase I of two phase method to solve this system Introducing artificial variables β1 and β2 the modified LPP become

Maximize $z' = -\beta_1 - \beta_2$

subject to

$$\begin{aligned}
2x_1 \quad + 2\lambda_1 + 2\lambda_2 - \mu_1 + \beta_1 + \quad &= 2 \\
3\lambda_1 + \lambda_2 - \mu_2 + \beta_2 \quad &= 1 \\
2x_1 + 3x_2 \quad s_1 \quad &= 6 \\
2x_1 + x_2 \quad s_2 &= 4
\end{aligned}$$

$$\mu_1 x_1 = 0,\ \mu_2 x_2 = 0,\ \lambda_1 s_1 = 0,\ \lambda_2 s_2 = 0$$

$$x_1, x_2, \lambda_1, \lambda_2, \mu_1, \mu_2, \beta_1, \beta_2, s_1, s_2 \geq 0$$

Initiat talle of Phase–I is

		C_j	0	0	0	0	0	0	–1	–1	0	0	
C_B	B.V	X_B	x_1	x_2	λ_1	λ_2	μ_1	μ_2	β_1	β_2	s_1	s_2	
–1	β_1	2	[2]	0	2	2	–1	0	1	0	0	0	→
–1	β_2	1	0	0	3	1	0	–1	0	1	0	0	
0	s_1	6	2	3	0	0	0	0	0	0	1	0	
0	s_2	4	2	1	0	0	0	0	0	0	0	1	
z′ =	–3		–2	0	–5	–3	1	1	0	0	0	0	
			↑										

Accoding to the regular pocedure λ_1 enters and β_2 leave the basis is $\lambda_1 > 0$ & $\beta_2 = 0$. But $s_1 = 6$ $\therefore \lambda_1 s1 \neq 0$.

$\therefore \lambda_1$ cannot enter the basis.

Next negative z; –e, is associated with λ_2. If λ_2 enters the basis then β_1 and β_2 will leave the basis is $\lambda_1 > 0$.

Since $s_2 = 4$ we have $\lambda_2 s_2 \neq 0$. So λ_2 cannot enter the basis.

Next negative z ; –e ; is associated with x_1. If x_1 eneris the basis then β_1, leaves the basis ie. $x_1 \geq 0$. This is accepted since $\mu_1 = 0$ & $\mu_1 . x_1 = 0$ is satisfied.

The next table is

		C_j	0	0	0	0	0	0	–1	–1	0	0
C_B	B.V	X_B	x_1	x_2	λ_1	λ_2	μ_1	μ_2	β_1	β_2	s_1	s_2
–1	x_1	1	1	0	1	1	–1/2	0	1/2	0	0	0
–1	β_2	1	0	0	3	1	0	–1	0	1	0	0
0	s_1	4	0	[3]	–2	–2	1	0	–1	0	1	0
0	s_2	2	0	1	–2	–2	1	0	–1	0	0	1
z′ =	–3		0	0	–3	–1	0	1	1	0	0	0
				↑								

Here λ_1 enters and β_1 leaves the basis i.e.. $\lambda_1 > 0$, $\beta_2 = 0$

This is not accepted since $s_1 = 4 \quad \therefore \lambda, s \neq 0$.

If λ_2 enters the basis then x_1 or β_2 leaves the basis..

This is not also accepted since $s_2 = 2$ & so $\lambda_2 s_2 \neq 0$

We select x_2 to enter the basis. Then s_1 leaves the basis.

The next table is

		C_j	0	0	0	0	0	0	−1	−1	0	0	
C_B	B.V	X_B	x_1	x_2	λ_1	λ_2	μ_1	μ_2	β_1	β_2	s_1	s_2	
0	x_1	1	1	0	1	1	−1/2	0	1/2	0	0	0	
−1	β_2	1	0	0	[3]	1	0	−1	0	1	0	0	→
0	x_2	4/3	0	1	−2/3	−2/3	1/3	0	−1/3	0	1/3	0	
0	s_2	2/3	0	0	−4/3	−4/3	2/3	0	−2/3	0	−1/3	1	
$z' =$	−1		0	0	−3	−1	0	1	1	0	0	0	
					↑								

Hare λ_1 enters the basis and β_2 leaves the basis. This is acceptable since $s_1 = 0 \quad \therefore \lambda_1 s_1 = 0$.

The next table is

		C_j	0	0	0	0	0	0	−1	−1	0	0
C_B	B.V	X_B	x_1	x_2	λ_1	λ_2	μ_1	μ_2	β_1	β_2	s_1	s_2
0	x_1	2/3	1	0	0	2/3	−1/2	1/3	1/2	−1/3	0	0
0	λ_1	1/3	0	0	1	1/3	0	−1/3	0	1/3	0	0
0	x_2	14/9	0	0	0	−4/9	1/3	−2/9	−1/3	2/9	1/3	0
0	s_2	10/9	0	0	0	−8/9	2/3	−4/9	−2/3	4/9	−1/3	1
$z' =$	0		0	0	0	0	0	0	1	1	0	0

In this table $\beta_1 = 0$ and $\beta_2 = 0$. So this is the final table.

The optimal solution is

$x_1 = 2/3, x_2 = 14/9, \lambda_1 = 1/3, \lambda_2 = 0, s_1 = 0, s_2 = 10/9, \mu_1 = 0, \mu_2 = 0$

The complementay stachness conditions

$\mu_1 x_1 = 0, \mu_2 x_2 = 0, \lambda_1 s_1 = 0$ & $\lambda_2 s_2 = 0$ are satisfied.

∴ The optimal solution of the given quadratic programming problem is

$x_1 = 2/3$, $x_2 = 14/9$

and $z_{man} = 2(2/3) | 14/9 - 2/3 = 22/9$

5.4 Beale's Method

Beale suggested another approach to solve quadratic programming problem (QPP)

Let the QPP be of the from

$$\text{Maximize } f(x) = cx + \frac{1}{2} x^T Q x$$

$$\text{subject to } Ax = b,\ x \geq 0$$

Where $x = [x_1, x_2, ... x_n]^T$, $C = [c_1, c_2, ...c_n]$, A is mxn matrix and Q is symmetric matrix.

In This method the variables are partitioned into basic and non-basic variables. At each iteration, the objective function is expressed in terms of te non-basic variables.

The Beale's iteative procedure of solving QPP is stated below :

Step 1. Express the constaints of the given QPP as equations by introducing slack / surplus variables to get $Ax = b$.

Step 2. Select arbitrarily m variables as basic and the remaining n–m variables as non-basic. With this partitioning, the constraint equation $Ax = b$ can be written as

$$[B \quad R] \begin{bmatrix} x_B \\ x_B \end{bmatrix} = b$$

or, $Bx_B + Rx_R = b$

Where x_B and x_R denote the basic and non-basic vectors respectively. Thus we get

$$x_B = B^{-1}b - B^{-1}Rx_R$$

Step 3. Express the basic x_B in terms of non-basic x_R only, using the given and additional constraint equations, if any.

Step 4. Express the objective function $f(x)$ in terms of xR only using the given and additional constraints, if. As $x_B \geq 0$ we have $B^{-1}RxR \leq B^{-1}b$. Thus, any component of xR can increase only until $\delta f/\delta x_R$ becomes zero, or one or more components of x_B are reduced to zero.

Note that we face the possibilit of having moer than m non-zero variables at any step of teration. This stage comes when the new point generated at some step occurs were $\delta f/\delta x_R$ becomes zero. Geometrically, this means that we are no longe at an extreme point of the convese set formed by the constaints, and thus no longer have a basic solution with respect to the original constraint set. When this happens, we simply define a new variables s_i as $s_i = \delta f/\delta x_{Ri}$ and a new constraint $s_i = 0$.

Step 5. At this stage, we have $m + 1$ non-zero varibles and $m + 1$ constraints, which is a basic solution to the extended set of constaints.

Step. Repeat the above procedure until no further improvement of the objective function may be obtained by increasing one of the non-basic variables.

Example 5.4.1. Using Beale's method solve the QPP

Maximize $z = 5 + 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$

subject to $x_1 + 2x_2 \leq 0$

$x_1, x_2 \geq 0$

Solution :

Introducing slacle variable $x_3 \geq 0$, the given QPP becomes

Maximize $z = 5 + 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$

subject to $x_1 + 2x_2 + x_3 = 2$

$x_1, x_2, x_3 \geq 0$

We choose x_1 arbitarily as basic variable and express it in terms of x_2 and x_3. Thus

$$x_1 = 2 - 2x_2 - x_3$$

We now express the objective functions z in terms of x_R $z = 5 + 4\,(2 - 2x_2 - x_3) + 6x_2 - 2\,(2 - 2x_2 - x_3)^2 - 2\,(2 - 2x_2 - x_3)\,x_2 - 2x_2^2$

$$\therefore \frac{\partial z}{\partial x_2} = -8 + 6 - 4\,(2 - 2x_2 - x_3)\,(-2) - 2\,(2 - 4x_2 - x_3) - 4x_2$$

At $x_2 = 0$m $x_3 = 0$ We have $\dfrac{\partial z}{\partial x_2} = -8 + 6 + 14 - 4 = 10$

This means z will increase if x_2 is increased from zero.

Also $\frac{\partial z}{\partial x_3} = -4 + 4\,(2 - 2x_2 - x_3) + 2x_2$

$\therefore$ At $x_2 = 0$, $x_3 = 0$ we have $\frac{\partial z}{\partial x_3} = -4 + 8 = 4$

We see that the rate of increase of z with respect to x_2 is more.

Hence incease in x_2 will give better improvement in the objective function.

To find how much x_2 should or may increase, we check two quantities.

(i) the value of x_2 for which $\delta z/\delta x_2$ vanishes.

(ii) the largest value of x_2 attained without deriving the basic variable x_1 negative.

Then x_2 will be minimum of these two.

Now $\delta z/\delta x_2 = 0$ gives for $x_3 = 0$

$$-2 + 8\,(2 - 2x_2) - 2\,(2 - 4x_2) - 4x_2 = 0$$

or, $-2 + 16 - 16x_2 - 4 + 8x_2 - 4x_2 = 0$

or, $-12x_2 + 10 = 0$

or, $x_2 = 5/6$

And for $x_3 = 0$, $x_1 < 0$ gives $2 - 2x_2 < 0$ or, $x_2 > 1$

We have min $\{5/6, 1\} = 5.6$. Thus the new basic variable is x_2.

Expressing x_2 is terms of x_1 and x_3 we get

$$x_2 = 1 - x_{1/2} - x_{3/2}$$

We now express z in terms of x_1 and x_3 as

$$z = 5 + 4x_1 + 6\,(1 - x_{1/2} - x_{3/2}) - 2x_1^2 - 2x_1\,(1 - x_{1/2} - x_{3/2}) - 2\,(1 - x_{1/2} - x_{3/2})^2$$

$$\text{Now } \frac{\partial z}{\partial x_1} = 4 - 6\,(-1/2) - 4x_1 - 2x_1\,(-1/2) - 2\,(1 - x_{1/2} - x_{3/2}) - 4\,(1 - x_{1/2} - x_{3/2})\,(-1/2)$$

$$= 1 - 3x_1$$

$$\frac{\partial z}{\partial x_3} = 6\ (1 - 1/2) - 2x_1\ (-1/2)\ \ 4\ (1 - x_{1/2} - x_{3/2})\ (-1/2)$$

$$= -1 - x_3$$

At $x_2 = 0$, $x_3 = 0$ We have $\frac{\partial z}{\partial x_1} = 1$ and $\frac{\partial z}{\partial x_3} = -1$

Ths z increases as x_1 is increases. So x_1 can be introduced to incease z.

To find how much x_1 should or may increase, we check two quantities.

(i) the value of x_1 for which $\delta z/\delta x_1$ vanishes.

(ii) the largest value of x_1 attained without deriving the basic variable x_2 negative.

The x_1 will be minimum of these two.

Fo $x_3 = 0$, $\delta z/\delta x_1 = 0$ gives $1 - 3x_1 = 0$ or $x_1 = 1/3$

For $x_2 = 0$, $x_2 < 0$ gives $1 - x_{1/2} < 0$ or, $x_1 > 2$

We have min $\{1/3, 2\} = 1/3$

Hence we find $x_1 = 1/3$ and the new basic vaiable is x_1.

At $x_1 = \frac{1}{3}$, $x_3 = 0$ we have $\frac{\partial z}{\partial x_1} = 0, \frac{\partial z}{\partial x_3} = -1$. Thus the optimal solution has been attained & the optimal solution is $x_1 = 1/3$, $x_2 = 1 - 1/6 - 0 = 5/6$, $x_3 = 0$ and man $x = 5 + 4/3 + 6 \times 5/6 - 2x\ (1/3)^2 - 2\ (1/3)\ (5/6) - 2x\ (5/6)^2 = 55/6$

5.5 Summary

Quadratic programming problem is concerned with non linear progamming problem of maximizing (or minimizing) the quadratic objective function subject to a set of linear inequality constaints. Wolfe's modified simplex method and Beale's method are discussed here with examples.

5.6 Self Assessment Questions

1. Applying wolfe's method solve the following quadatic pogramming problems

(i) Maximize $f = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$

subject to $x_1 + 2x_2 \leq 2$

$x_1, x_1 \geq 0$

(ii) Maximize $z = 12x_1 + 12x_2 - 18x_1^2 - 12x_1x_2 - 8x_2^2$

subject to $3x_1 + 4x_2 \leq 2$

$x_1, x_1 \geq 0$

(iii) Maximize $f = 3x_1 + 2x_2 - 2x_2^2$

subject to $4x_1 + x_2 \leq 4$

$2x_1 + x_2 \leq 2$

$x_1, x_2 \geq 0$

(iv) Maximize $z = 10x_1 + 6x_2 - 50x_1^2$

subject to $5x_1 + 8x_2 \leq 4$

$5x_1 + 4x_2 \leq 2$

$x_1, x_2 \geq 0$

(iv) Maximize $f = -4x_1 + x_1^2 - 2x_1x_2 + 2x_2^2$

subject to $2x_1 + x_2 \leq 6$

$x_1 - 4x_2 \leq 0$

$x_1, x_2 \geq 0$

(iv) Maximize $z = 2x_1 + 3x_2 - 2x_1^2$

subject to $x_1 + 4x_2 \leq 4$

$x_1 + x_2 \leq 2$

$x_1, x_2 \geq 0$

2. Use Beale's method of solve the following quadiratic linear programming problems

(i) Maximize $z = 6 - 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2$

subject to $x_1 + x_2 \le 2$

$x_1, x_2 \ge 0$

(ii) Maximize $z = 2x_1 + 3x_2 - x_1^2$

subject to $x_1 + 2x_2 \le 4$

$x_1, x_2 \ge 0$

(iii) Maximize $f = 2x_1 + 3x_2 - 2x_2^2$

subject to $x_1 + 4x_2 \le 4$

$x_1 + x_2 \le 2$

$x_1, x_2 \ge 0$

(iv) Maximize $f = 12x_1 + 6x_2 - 18x_1^2 - 6x_1x_2 - 2x_2^2$

subject to $3x_1 + 2x_2 \le 2$

$x_1, x_2 \ge 0$

Unit 6 ❑ Integer Programming Problem

Structure

6.1 Introduction

Integer Programming Problem (IPP) is a special class of Linear Programming Problem where all or some of the variables in the optimal solution are restrieted to the integers. If all the variables are restricted to take integral values the IPP is termed as pure IPP. On the other hand, if only some variables are restricted to take only integer values then the problem is called mixed IPP.

In 1956, R. E. Gomory developed a method to solve pure IPP. Later, he extended the method to solve mixed IPP. Another important approach, called the "branch and bound" technique was developed for solving both the all integer and he mixed integer programming problems.

Several algorillms have yet been developed for solving both types of IPP. We shall discuss only.

(i) Gomory's cutting plane method for pure IPP. and

(ii) Branch and bound method.

6.2 Need for Integer Programming

To solve an IPP one may think to get the optimal solution just by rounding down the optimal solution of the corresponding LPP obtained by regular simplex method. But there is no gaurantee for this. It may or may not happen so. The integer solution obtaind by rounding down the optimal solution of the corresponding LPP will not always satisfy all constraints or will not give the actual optimal solution of the IPP. These are explained by following examples.

Example 6.2.1

Maximize $z = 3x_1 - 2x_2$

subject to $12x_1 + 7x_2 \leq 28$

$x_1, x_2 \geq 0$

x_1, x_2 are integers.

Ignoring the integer restriction here the optimal solution is $x_1 = 2\frac{1}{3}$, $x_2 = 0$ with max $z = 7$.

The solution obtained by rounding down this optimal solution is $x_1 = 2$, $x_2 = 0$ this solution is the optimal solution of the given Integer programming problem.

Example 6.2.2

Minimize $z = 2x_1 + 3x_2$

subject to $80x_1 + 31x_2 \geq 248$

$x_1, x_2 \geq 0$, x_1, x_2 are integers.

Here, ignoring the integer restriction, the optimal solution is $x_1 = 3\frac{1}{10}$, $x_2 = 0$ with min $z = 6\frac{1}{5}$

Rounding down the solution we get $x_1 = 3$, $x_2 = 0$

But this point does not lie in the feasible region since $80 \times 3 + 31.0 = 240 < 248$.

Hence just rounding the optimal solution of the corresponding LPP to the given IPP we may not get the optimal solution of the IPP.

Example 6.2.3

Maximize $z = 3x_1 + 4x_2$

subject to $4x_1 + 6x_2 \leq 15$

$x_1, x_2 \geq 0$

x_1, x_2 are integers.

Ignoring the integer-valued restriction The optimal solution of the problem is $x_1 = 3\frac{3}{4}$, $x_2 = 0$ with max $z = 11\frac{1}{4}$

Rounding off this solution we get $x_1 = 3$, $x_2 =$ or, $x_1 = 4$, $x_2 = 0$.

For $x_1 = 3$, $x_2 = 0$ we have $z = 3 \times 3 + 4 \times 0 = 9$

$x_1 = 4$, $x_2 = 0$ does no satisfy $4x + 6x_2 \leq 15$. Here the actual solution tothis IPP is $x_1 = 2$, $x_2 = 1$ with max $z = 10$.

6.3 Gomory's cutting plane method for all IPP

In this method we first find the optimal solution to the IPP by simplex method ingoring the integer valued restriction. If in the optimal solution all the variables have integer values, then it is also the optimum solution of the given IPP. But if not, then a new constraint, called secondary an Gomory's constraint is introduced to the problem which slice away non-integer optimal solution exhibited by the extreme point of the feasitle region of the associated LPP and at the same time leave all feasible integer solutions untouched. The new related LPP is then solved as usual. If the new optimal solution obtained does not satisfy the integer requirement, then another Gomory's constraint is added and the process is repeated iteratively until the required integer valued optimum solution is obtained. As each introduced Gomory's constraint cut off a portion of the feasible region of the related LPP, the method is called Gomory's cutting plane method.

6.3.1 Construction of Gomory's constraints

Ignoring the integer restriction let the optimal solution of the given IPP using simplex method be x_B. Also let this optimal solution has at least one non-integer component. If more than one basic variable are fractional, we select that non-integral variable which involves the largest fractional part.

As x_{Br} corresponds to the rth now of simplex table we consider the rth now. of the final tables as

$$\sum_{j=1}^{n} y_{rj} x_j = b_r \quad .... \quad .... \quad .... (1)$$

Let $[y_{rj}]$ denote the greatest integer less than y_{rj} and fri denote the positive fractional part of y_{rj}. Similarly, let $[b_r]$ and f_r be resputively the greatest integer less than b_r and the positive fractional part of br.

Then we have $y_{rj} = [y_{rj}] + f_{rj}$

and $b_r = [b_r] + f_r$ where $0 < f_{rj} < 1$ and $o < f_r < 1$.

From (1) we have thus

$$\sum_{j=1}^{n} [y_{rj}] x_j + \sum_{j=1}^{n} f_{rj} x_j = [b_r] + f_r$$

and $$f_r - \sum_{j=1}^{n} f_{rj} x_j = [b_r] - \sum_{j=1}^{n} [y_{rj}] x_j \quad .... \quad .... \quad .... (2)$$

For integer value of x_j the RHS of (2) is an integer. So LHS of (2) must be an integer. Now f_r is a proper fraction *i.e.* $0 < f_r < 1$ and $\sum_{j=1}^{n} f_{rj} x_j$ is positive thus (2) gives.

(A proper fraction) – (positive number) = (integer)

Hence RHS is either zero or negative integes.

So LHS is also either zero or negative integer

i.e. LHS ≤ 0

or, $$\sum_{j=1}^{n} f_{rj} x_j \leq 0$$

or, $$-\sum_{n=1}^{n} f_{rj} x_j \leq -f_r$$

Introducing plack variable x_s this becomes

$$-\sum_{n=1}^{n} f_{ij}x_j + x_s = -f_i$$

This is the Gomory's constraints which is to be introduced to the given problem to form a new LPP to be solved the dual simplex method.

6.3.2 Gomory's cutting Plane Algorithm

The following are the four steps of solving all integer IPP by Gomory's cutting plane method.

Step 1. Using simplex method find the optimal solution of the IPP ignoringthe integral value restructions.

Step 2. If all the variables have integral values, take this solution as the optimal solution of the given IPP.

If at least one varible in the optimal solution obtained in setp 1 has fractional value then identify the now involving the largest fractional part. Using this row from the Gomory's constraint.

Step 3. Augment the IPP by introducing the Gomory's constraint formed in step 2 and modify the table. Using dual simplex method find the new optimal solution of the augmented LPP.

Step 4. If all variables of the optimal solution obtained in setp 3 are integers, then this is the required optimal solution of the original IPP. Otherwise go to step 2 and again augment the IPP by a new Gomory's constraint.

Example 6.3.1 Use Gomory's cutting plane method to find the optimal solution of the IPP

Maximize $z = x_1 + x_2$

subject to $2x_1 + 5x_2 \le 16$

$6x_1 + 5x_2 \le 30$

$x_1, x_2 \ge 0$

x_1, x_2 are integers.

Solution : Ignoring the intergal value restriction we solve itby simplex method. Introducing slack variables x_3 and x_4 the LPP becomes

Maximize $z = x_1 + x_2 + 0x_3 + 0x_4$

subject to $2x_1 + 5x_2 + x_3 = 16$

$6x_1 + 5x_2 + \quad + x_4 = 30$

$x_1, x_2, x_3, x_4 \geq 0$

Using simplex method the tables are obtained

		c_j	1	1	0	0	
C_B	y_B	x_B	y_1	y_2	y_3	y_4	min ratio
0	y_3	16	2	5	1	0	8
0	y_4	30	[6]	5	0	1	5 →
z = 0			–1	–1	0	0	
0	y_3	6	0	[10/3]	1	–1/3	9/5 →
1	y_1	5	1	5/6	0	1/6	6
z = 5			0	–1/6	0	1/6	
1	y_2	9/5	0	1	3/10	–1/10	
1	y_1	7/2	1	0	–1/4	1/4	
z = 53/10			0	0	1/20	3/20	

In this object table we see that both the variables are fractional and are 9/5 = 1 + 4/5, 7/2 = 3 + 1/2. The largest fractional part is 4/5 and is associated with the first row. The first row written in the form of equation is

$x_2 + (3/10)\ x_3 - (1/10)\ x_4 = 9/5$

Writing 3/10 = 0 + 3/10, –1/10 = –2 + 9/10 and 9/5 = 1 + 4/5 this becomes

$x_2 + 0x_3 + (3/10)\ x_3 - 2x_4 + (9/10)\ x_4 = 1 + 4/5$

∴ The Gomory's constraint is

$-\ (3/10)\ x_3 - (9/10)\ x_4 - \leq (4/5)$

Introducing plack variable $x_5 \leq 0$ we get

$-\ (3/10)\ x_3 - (9/10)\ x_4 + x_5 = -\ (4/5)$

Adding this Gomory's constraint to the above optimum table, we get modified table as follows :

		c_j	1	1	0	0	0
C_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
1	y_2	9/5	0	1	3/10	–1/10	8
1	y_1	7/2	1	0	–1/4	1/4	0
0	y_5	–4/5	0	0	[–3/10]	–9/10	1
		$z_j - c_j$	0	0	1/20	3/20	1
$\frac{(z_j - c_j)}{y_{3j}}$: $y_{3j} < 0$					$\frac{1/20}{-3/10}$	$\frac{3/20}{-9/10}$	
1	y_2	1	0	1	0	-1	1
1	y_1	25/6	1	0	0	1	–5/6
0	y_3	8/3	0	0	1	3	–10/3
		$z_j - c_j$	0	0	0	0	1/6

In this optimal table the basic variable x_1 is fractional (it is a variable of the original given IPP). It is associated with second row. We consider the second row and write it as equation to form Gomory's second constraint.

$x_1 + x_5 \;\; - (5/6)\, x_5 = 25/6$

or, $x_1 + x_5 + (-1)\, x_5 + (1/6)\, x_5 = 4 + 1/6$.

The Gomory's constraint is

$- (1/6)\, x_5 \leq - (1/6)$

or, $-x_5 \leq - 1$

Adding slack variable x_6 0 we get

$- x_5 + x_6 = - 1$

Adding the Gomory's constraint to the above optimum table and modifying the table we get

		c_j	1	1	0	0	0	0
C_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
1	y_2	1	0	1	0	–1	1	0
1	y_1	25/6	1	0	0	1	–5/6	0
0	y_3	8/3	0	0	1	3	–10/3	0
0	y6	–1	0	0	0	0	[–1]	1
		$z_j - c_j$	0	0	0	0	1/6	0
$\frac{(z_j - c_j)}{y_{3j}}$	$: y_{4j} < 0$						$\frac{1/6}{(-1)}$	
1	y_2	0	0	1	0	–1	0	1
1	y_1	5	1	0	0	1	0	–5/6
0	y_3	6	0	0	1	3	0	–10/3
	y_5	1	0	0	0	0	1	–1
		$z_j - c_j$	0	0	0	0	1/6	

As the original variables are integers this is the final table of the IPP. The optimal solution is $x_1 = 5$, $x_2 = 0$ and max $z = 5$.

6.4 The Branch and Bound Method

The Branch and Bound method is most powerful method and is applicable to both pure as well as mixed integer programming prolbems. This method was developed by Landand Doig. The principal idea underlying the branch and bound method is an follows. First we are to solve the problem ignoring the integer valued restriction. If the optimal solution has non-integral value, say x_j, then there is an integer k such that $k < x_j < k + 1$. As we want x_j to have integer value, the value of x_j must satisfy either $x_j \leq k$ or xj $\geq k + 1$ but not noth. Adding these constraints individually to the constraints of the given problem two subproblems are obtained. These two subproblems are solved. Repating the branching, the desired optimal solution is obtained.

6.4.1 "ranch and Bound Algorithm

The step by step procedure of branch and bound algorithm is as follows :

Let the IPP be

Maximize $z = cx$

subject to $Ax = b$

$x \geq 0$

x_j is integer for $j \in I$

Where $c = [c_1, c_2, ..., c_n]$, $x = [x_1, x_2, ..., x_n]^T$, $b = [b_1, b_2, ..., b_m]^T$

$A = [a_{ij}]_{m \times n}$

If $I = \{1, 2, ..., n\}$ then it is a pure (or all) IPP and if I is a proper subset of $\{1, 2, ..., n\}$ then it is a mixed IPP.

Step 1. Ignoringthe integer restriction solve the IPP. If the optimal solution be such that all x_j, $j \in I$ are integers, then this is the required optimal solution. If at least one x_j, $J \in I$ be non-integer then go to next step.

Step 2. Among hon-integer x_j, $j \in I$ chope any one, Then there exists integer k such that

$$K < x_j < k + 1$$

As we want x_j to be an integer, the integer solution must satisfy one of the following

$$x_j \leq k \text{ or } x_j \geq k + 1$$

Add these constraints indirectly to the constraints of the current problem and get two sub-problems. Solve these two sub-problems.

Step 3. If for any of the subproblem integer solution is obtained then that problem is not further branched.

But if any subproblem involves some non-integer variable, then it is again branched. This process of branching is continued, until each subproblem either admits an integer valued solution or there is eirdence that it cannot yield a better solution or it gives no feasible solution.

Among all subproblems select that integer valued solution which gives the over all maximum value of the object function.

Note : Main disadvantage of this method is that it requires the optimal solution of each subproblem. For large size problem this become very tedions job. Inspite of this drawback it is most effective method for solving IPP. Also the method is applicable for both all and mixed IPP

Example 6.4.1 Using Branch and Bound technique solve the following IPP

Maximize $z = x_1 + x_2$

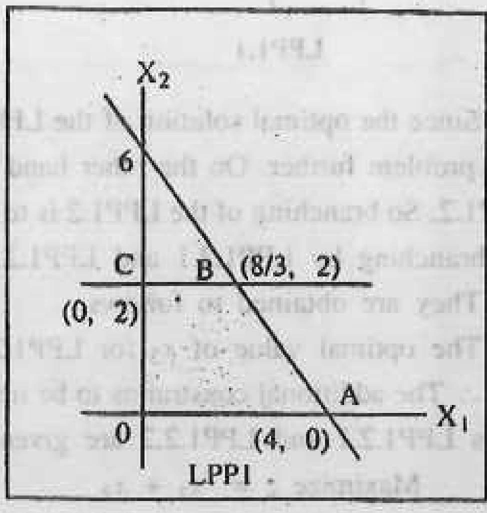
subject to $3x_1 + 2x_2 \le 12$ .... (LPP1)

$x_1, x_2 \le 0$

$x_1, x_2 \ge 0$

x_1, x_2 are integers.

Solution : Ignoring the integer valued restriction the solution of the given IPP by graphical method is $x_1 = 8/3, x_2 = 2$, the value of z is $4\frac{2}{3}$. We call the LPP corresponding to this IPP as LPP1.



The value of x_1 is fraction and is 8/3. We note that $2 < 8/3 < 3$.

So we from two subproblems with additional constraints respectively as $x_1 \le 2$ and $x_1 \ge 3$.

Thus two problems are

Maximize $z = x_1 + x_2$

subject to $3x_1 + 2x_1 \le 12$

$x_2 \le 2$ .... (LPP1.1)

$x_1 \le 2$

$x_1, x_2 \ge 0$

and

Maximize $z = x_1 + x_2$

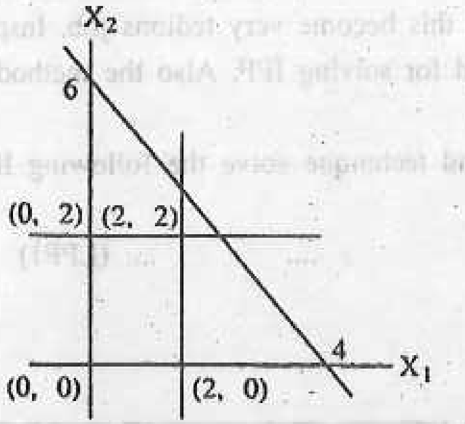
subject to $3x_1 + 2x_1 \le 12$

$x \le 2$ .... (LPP1.2)

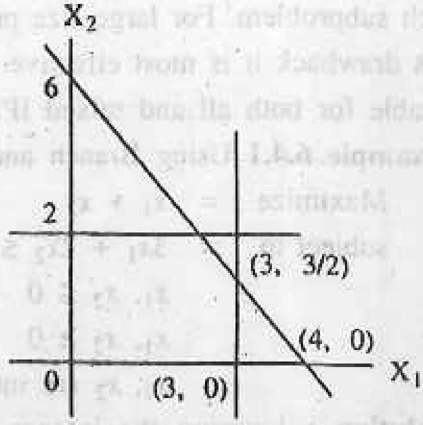
$x_1 \le 3$

$x_1, x_2 \ge 0$

By graphical method, the optimal solution of the LPP1.1 is $x_1 = 2$, $x_2 = 2$ with $z = 4$ and that of the LPP1.2 is $x_1 = 3$, $x_2 = 3/2$ with $z = 9/2$



LPP1.1



LPP1.2

Since the optimal solution of the LPP1.1 are integers there is no need to branch this problem further. On the other hand the optimal value of x_2 is fraction for the LPP1.2. So branching of the LPP1.2 is to be done. Let the two subproblems obtained by branching by LPP1.2.1 and LPP1.2.2.

They are obtained to follows.

The optimal value of x_2 for LPP1.2 is 3/2 and $1 < 3/2 < 2$.

∴ The additional constraints to be introduced are $x_2 \leq 1$ and $x_2 \geq 2$ respectively.

Thus LPP1.2.1 and LPP1.2.2 are given by

Maximize $z = x_1 + x_2$

subject to $3x_1 + 2x_1 \leq 12$

$x_2 \leq 2$

$x_1 \geq 3$ .... (LPP1.2.1)

$x_2 \leq 1$

$x_1, x_2 \geq 0$

and

Maximize $z = x_1 + x_2$

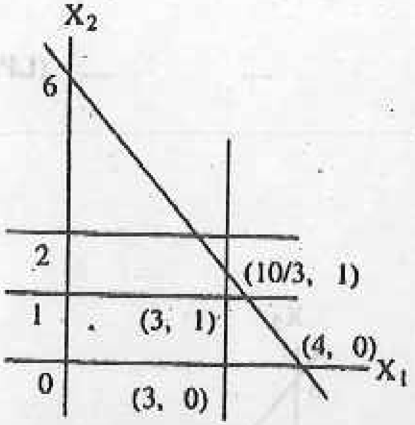
subject to $3x_1 + 2x_1 \leq 12$

$x_2 \leq 2$

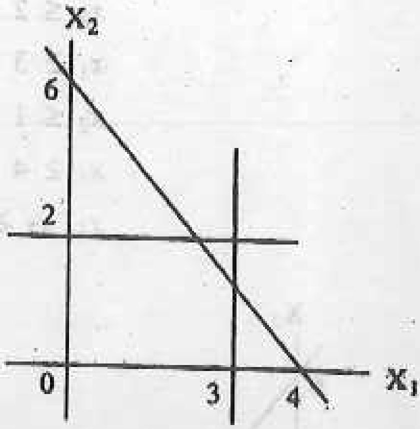
$x_1 \geq 3$

$x_2 \geq 2$ .... (LPP1.2.2)

$x_1, x_2 \geq 0$



LPP1.2.1



LPP1.2.2

Using graphical method the optimal solution of the LPP1.2.1 is $x_1 = 10/3$, $x_2 = 1$ with the value of $z = 13/3 = 4\frac{1}{3}$. As x_1 is not an integer and $z = 13/3$ which is greater that the optimal value $z = 4$ of the LPP1.1, we need branching of this LPP toget LPP1.2.1.1. and LPP1.2.1.2. (Here we note that instead of z = 13/3 if the value of z would be less than 4 then no branching is nedded)

The LPP1.2.2. has no feasible, so no question of branching.

To get branching of LPP1.2.1. we note that 3 < 10/3 < 4. So that additional constraints to the LPP.1.2.1 to get sub problem are respectively $x_1 \leq 3$ and $x_1 \geq 4$.

Thus the subproblems are given by

Maximize $z = x_1 + x_2$

subject to $3x_1 + 2x_1 \leq 12$

$x_2 \leq 2$

$x_1 \geq 3$ .... (LPP1.2.1.1)

$x_2 \leq 1$

$x_1 \leq 3$

$x_1, x_2 \geq 0$

and

Maximize $z = x_1 + x_2$

subject to $3x_1 + 2x_1 \leq 12$

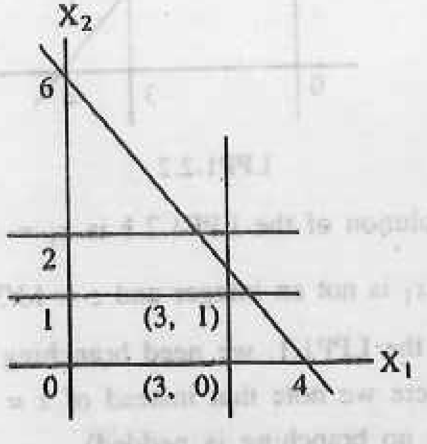
$x_2 \leq 2$

$x_1 \geq 3$ .... (LPP1.2.1.2)

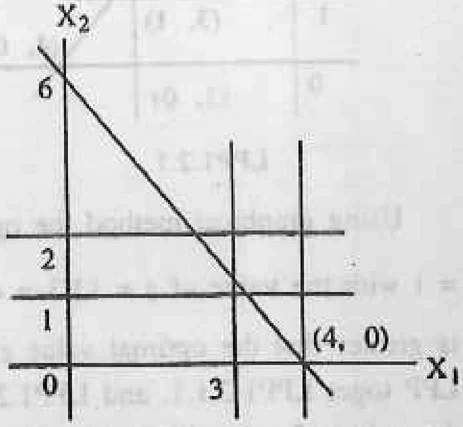
$x_2 \leq 1$

$x_1 \geq 4$

$x_1, x_2 \geq 0$



LPP1.2.1.1



LPP1.2.1.2

Graphical we get the optimal solution of the LPP1.2.11 as $x_1 = 3$, $x_2 = 1$ with $z = 4$ which is some as the optimal value of z of the LPP1.1. The optimal solution of the LPP 1.2.1.2. is $x_1 = 4$, $x_2 = 0$ with $z = 4$. No further branching is necessary.

The over all maximum value of the objective function is $z = 4$ and the integer valued solution are $x_1 = 2$; $x_1 = 3$, $x_2 = 1$; $x_1 = 4$, $x_2 = 0$.

6.5 Summary

Gomory cutting plane method for all IPP and Branch and bound method for general IPP have been considered and explained with examples. Need forIPP has been explained in detail with examples.

6.6 Self Assessment Questions

1. Solve the following IPP using Gomory's cutting plane method.

(i) Maximize $z = 2x_1 + 2x_2$

subject to $5x_1 + 3x_2 \leq 8$

$x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0$

x_1, x_2 are integers

[Ans : $x_1 = 1, x_2 = 1$, max $z = 4$]

(ii) Maximize $z = 4x_1 + 3x_2$

subject to $3x_1 + 4x_2 \leq 12$

$4x_1 + 2x_2 \leq 9$

$x_1, x_2 \geq 0$

x_1, x_2 are integers

[Ans : $x_1 = 1, x_2 = 2$, max $z = 10$]

(iii) Maximize $z = x_1 - 2x_2$

subject to $4x_1 + 2x_2 \leq 15$

$x_1, x_2 \geq 0$

x_1, x_2 are integers

[Ans : $x_1 = 3, x_2 = 0$, max $z = 3$]

2. Using Branch and Bound method solve the following IPP

(i) Maximize $z = 3x_1 + 4x_2$

subject to $3x_1 + 2x_2 \leq 8$

$x_1 + 4x_2 \leq 0$

$x_1, x_2 \geq 0$

x_1, x_2 are integers

[Ans : $x_1 = 1, x_2 = 1$, max $z = 11$]

(ii) Maximize $z = 7x_1 + 9x_2$

subject to $-x_1 + 3x_2 \leq 6$

$7x_1 + x_2 \leq 35$

$0 \leq x_1 \leq 7$

$0 \leq x_2 \leq 7$

x_1, x_2 are integers

[Ans : $x_1 = 4, x_2 = 3$, max $z = 55$]

Unit 7 ❑ One dimensional minimization method

Structure

7.1 Introduction

Numerical method ofoptimization are used to solve the problems involving objective function and/or constraints which are two complicated or cannot be expressed as explicit function.

One dimensional minimization method plays an important role to solve the problems using numerical technique. In numerical methods we are to minimize $f(x_i + \lambda_i S_i)$ with respect to li for known values of x_i and s_i.

This is nothing but aone dimensional minimization problem. Among many one-dimensional minimization methods Fibonacci method and golden section method are simple and important. They are discussed in this unit. These two methods are used for unimodal functions.

7.2 Unimodal Function

In the process of finding optimal point often it becomes necessary that the function has only one optimum point in the domain of search. As in many methods we need only the values of the function at various points, the functionmay not be continuous and differentiable. What we need is that it should be unimodal. Unimodality of a function of one variable is defined as follows

7.2.1. Definition

A real valued function f(x) is said to be unimodal (minimum) is $[a, b]$ if there is a point $x^* \in [a, b]$ such that

(i) it $a < x_1 < x_2$ x^* then $f(x_1) > f(x_2)$

(ii) it $a < x_1 < x_2 < b$ then $f(x_2) > f(x_1)$

7.3 Fibonacci Method

Fibonacci method is based on Fibonacci sequence (Fn) defined by

$$F_0 = F_1 = 1$$

$$F_n + F_{n-1} + F_{n-2},\ n = 2, 3, 4, \ldots\ldots\ldots$$

Thus

$F_0 = 1$, $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5$, $F_5 = 8$, $F_6 = 13$, $F_7 = 21$, $F_8 = 34$, $F_9 = 55$, $F_{10} = 89$, $F_{11} = 144$, ...

Fibonacci method can be used to find the optimum of a function of one variable. The function must be unimodal, it may ormay not be continuous or differtiable. This method has the following limitations :

(i) The initial interval of uncertainty $[a, b]$, in which the optimum lies, has to be known

(ii) The function to be optimized has to be unimodal in the initial interval of uncertainty.

(iii) The exact optimum point cannot be located by this method. Only an interval, known as the final interval of uncertainty can be obtained.

(iv) The number of function evaluations to be used in the search has to be specified beforehand.

The final interval of uncertainly can be made as small as we desire by making the number of function evaluations more.

Procedure : Let L be the length of the initial interval of uncertainty $[a, b]$ be the initial interval of uncertanity. Therefore $L_0 = b - a$.

Let n be the total number of experiments to be conducted. We define

$$L_2^* = \frac{F_{n-2}}{F_n} L_0$$

The first two experiments are placed at the points x_1 and x_2 which are located at a distance L_2^* from each end of L_0. The values of the function f at x_1, x_2 are evaluated as $f_1 = f(x_1)$ at $f_2 = f(x_2)$. Using unimodality assumption one of the intervals $[a, x_1]$ and $[x_2, b]$ is to be discarded. The remaning interval of uncertainty is denoted by L_2.

Then $L_2 = L_0 - L_2^*$

$$= L_0 - \frac{F_{n-2}}{F_n} . L_0$$

$$= L_0\left(\frac{F_n - F_{n-2}}{F_n}\right)$$

$$= \frac{F_{n-1}}{F_n} L_0$$

Now $L_2 - L_2^*$

$$= \frac{F_{n-1}}{F_n} L_0 - \frac{F_{n-2}}{F_n} L_0$$

$$= \frac{L_0}{F_n}(F_{n-1} - F_{n-2})$$

$$= \frac{L_0}{F_n}(F_{n-2} + F_{n-3} - F_{n-2})$$

$$= \frac{F_{n-3}}{F_n} L_0$$

$$\therefore \quad \frac{L_2 - L_2^*}{L_2^*} = \frac{F_{n-3}}{F_{n-2}} < 1$$

i.e. $L_2 - L_2^* < L_2^*$

Thus in the interval of uncertainty L2 there is one point, either x_1 or x_2, whose distance from the two ends of L_2 are L_2^* and $L_2 - L_2^*$. The smaller of the two $L_2 - L_2^*$ & L_2^* is denoted) by. i.e L_3^*. $L_3^* = L2 - L_2^*$

Now, $L_3^* = L_2 - L_2^* = \frac{F_{n-3}}{F_n} L_0$.

We now place the third experiment x_3 and L_2 so that the current two experiment are located at a distance L_3^* from each end of L_2. Again by the unimodal property we can reduce the interval of uncertainty from L_2 to L_3 given by $L_3 = L_2 - L_3^* = \frac{F_{n-2}}{F_n} L_0$.

∴ The interval of uncertainty at the end of 3rd experiment is given by

$$L_3 = \frac{F_{n-2}}{F_n} L_0$$

and this obtained by discarding $L_3^* = \frac{F_{n-3}}{F_n} L_0$ continuing in this manner we have the following result in general.

The j th experiment is to be placed at a distance $L_j^* = \frac{F_{n-j}}{F_n} L_0$ from one end of L_{j-1} and the interval of uncertainty at the end of j th experiment is given by $L_j = \frac{F_{n-j+1}}{F_n} L_0$

Taking j = n we see that the nth experiment is to be placed at a distance L_n^*

$= \frac{F_0}{F_n} L_0 = \frac{L_0}{F_n}$ from one end of L_{n-1} and the interval of uncertainty at the end of n th experiment is given by $L_n = \frac{F_1}{F_n} L_0 = \frac{L_0}{F_n}$

$$\text{Now } L_{n-1} = \frac{F_{n-(n-1)+1}}{F_n} L_0 = \frac{F_2}{F_n} L_0 = \frac{2L_0}{F_n}$$

$$\therefore L_n^* = \frac{1}{2} L_{n-1}$$

Therefore, the last two experiments are located at a distance $L_n^* = \frac{1}{2} L_{n-1}$ from each end of L_{n-1}. So they have the same location. To remove this difficulty we place the n th experiment very close to the remaining valid experiment in L_{n-1}. This enables us to obtain the final interest of uncertainty of length $\frac{1}{2} L_{n-1} = L_n = \frac{L_0}{F_n}$

From $L_n = \frac{L_0}{F_n}$ we note that we can determine n for given L_n

7.4 Illustrative Examples

Example 7.4.1 : Maximize $f(x) = \begin{cases} 2x/3, & x \le 3 \\ 5 - x, & x > 3 \end{cases}$

in the interest [1, 4] by Fibonacci method using $n = 6$

Solution : Here number it experiment to be performed is $n = 6$.

From Fibonacei sequence we have

$F_0 = F_1 = 1$

$F_2 = 2$, $F_3 = 3$, $F_4 = 5$, $F_5 = 8$, $F_6 = 13$, $F_7 = 21$ etc.

Here $L_0 = 4 - 1 = 3$.

$$\therefore L_2^* = \frac{F_4}{F_6} L_0 = \frac{5}{13} \times 3 = 1 \cdot 1538$$

The first two experiments are placed at the positions x_1 and x_2 such that

$$x_1 = 1 + L_1^* = 1 + 1{\cdot}1538 = 2{\cdot}1538$$

& $\quad x_2 = 4 - L_1^* = 4 - 1{\cdot}1538 = 2{\cdot}8462$

Now $f_1 = f(x_1) = \frac{2x_1}{3} = \frac{2 \times 2 \cdot 1538}{3} = 1 \cdot 4359$

and $f_2 = f(x_2) = \frac{2x_2}{3} = \frac{2 \times 2 \cdot 8462}{3} = 1 \cdot 8975$

Since $f_1 < f_2$, using uniondal property we delete the interval $[1, x_1]$. Thus the reduced interval of uncertainly is $[x_1, 4]$ i.e., $[1{\cdot}4359, 4]$ with x_2 inside it and near to x_1.

The third experiment is placed at the position x_3 given by

$$4 - x_3 = x_2 - x_1$$

or, $\quad x_3 = 4 - x_2 + x1$

$$= 4 - 2{\cdot}8462 + 2{\cdot}1538$$

$$= 3{\cdot}3076$$

Now $f_3 = f(x_3) = 5 - x_3 = 5 - 3{\cdot}3076 = 1{\cdot}6924$

Here $f_3 < f_2$. So by unimodaily we delete the interval $[x_3, 4]$. The remaining interval of uncertainly becomes $[x_1, x_3]$ with x_2 inside it and near to the point x_3.

The fourth experiment is placed at x_4 given by

$$x_4 - x_1 = x_3 - x_2$$

$\therefore\ x_4 = x_1 + x_3 - x_2 = 2{\cdot}1538 + 3{\cdot}3076 - 2{\cdot}8462 = 2{\cdot}6152$

Now, $f_4 = (x_4) = \frac{2x_4}{3} = \frac{2 \times 2 \cdot 1652}{3} = 1 \cdot 7435$

Since $f_4 < f_2$ we delite the interval $[x_1, x_4]$. The remaining interval of uncertainly is $[x_4, x_3]$ with x_2 inside it and near to x_4.

The fifth experiment is placed of x_5 given by

$$x_3 - x_5 = x_2 - x_4$$

or, $x_5 = x_3 - x_2 + x_4 = 3{\cdot}3076 - 2{\cdot}8462 + 2{\cdot}6152 = 3{\cdot}0766$

Now $f_5 = f(x_5) = 5 - x_5 = 5 - 3{\cdot}0766 = 1{\cdot}9234$

Since $f_5 < f_2$, using unimodal property we delete the interval $[x_4, x_2]$. The remaining interval of uncertainty is $[x_2, x_3]$ with x_5 inside it and near x_2.

The sixth experiment is placed at x_6 given by

$$x_3 - x_6 = x_5 - x_2$$

or, $x_6 = x_3 - x_5 + x_2 = 3{\cdot}3076 - 3{\cdot}0766 + 2{\cdot}8462 = 3{\cdot}0772$

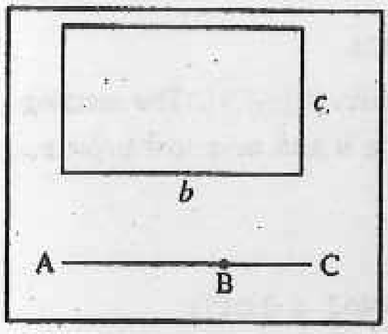
Now $f_6 = f(x_6) = 5 - x_6 = 5 - 3{\cdot}0772 = 1{\cdot}9228$

since $f_6 < f_5$, using unimodality we delete the interval $[x_6, x_3]$. The final interval of uncertainty is $[x_2, x_6] = [2{\cdot}8462, 3{\cdot}0772]$

Here we note that if the exact calculation be carried out then we would get $x_5 = x_6$. In that situation x_6 should be selected very close to x_5. But here we see $x_5 \neq x_6$. This is due to round off error involved in the calcution.

7.5 Golden Section

Ancient Greek architects believed that a building having sides b and c satisfying the relation $\frac{b+c}{b} = \frac{b}{c} = \gamma$ will be having the most pleasing properties. This ratio is called Golden ration. It is also found in Euclid's geometry that the division of a line segment into unequal parts so that the ration of the whole to the largest part is equal to the ratio of the large part to the smaller part: This section is known as the golden section



Thus the Golden section

$$\frac{AC}{AB} = \frac{AB}{BC} = \gamma \quad \text{i.e.,} \quad \frac{AB + BC}{AB} = \frac{AB}{BC} = \gamma$$

From this we have

$$\frac{AB}{AB} + \frac{BC}{AB} = \frac{AB}{BC} = \gamma$$

or, $1+\frac{1}{\gamma}=\gamma$

or, $\gamma^2 - \gamma - 1 = 0$

$$\therefore \quad \gamma = \frac{-(-1) \pm \sqrt{(-1)^2 - 4.1.(-1)}}{2.1}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

Since γ is a positive number we have

$$\gamma = \frac{\sqrt{5}+1}{2} = 1 \cdot 618$$

7.6 Golden Section Method

Golden section method is similar to the Fibonacei method except for one difference. The difference is that in Fibonacei method the total number of experiments to be performed has to be specified before beginning the calculation, whereas, this is not required in golden section method. In fact when n is very large then Fibonacei method reduces to golden section method. In Fibonacei method the number of experiments to be performed is decided at the begining but in golden section method the total number of experiments are to be decided during the computations.

In the Fibonacei method, the interval of uncertanity at the end of two experiments is given by $L_2 = \frac{F_{n-1}}{F_n} L_0$

In Golden Section method is n is very large this L_2 becomes

$$L_2 = \lim_{n\to\infty} \frac{F_{n-1}}{F_n} L_0 = L_0 \left(\lim_{n\to\infty} \frac{F_{n-1}}{F_n} \right)$$

Also in Fibona method L_3 is given by

$$L_3 = \frac{F_{n-2}}{F_n} L_0$$

∴ In Golden section method L_3 will be given by

$$L_3 = \lim_{n\to\infty} \frac{F_{n-1}}{F_n} L_0$$

$$= \lim_{n\to\infty} \left(\frac{F_{n-2}}{F_{n-1}} \cdot \frac{F_{n-1}}{F_n} L_0 \right)$$

$$= L_0 \cdot \left(\lim_{n\to\infty} \frac{F_{n-2}}{F_{n-1}} \right) \left(\lim_{n\to\infty} \frac{F_{n-1}}{F_n} \right)$$

$$= L_0 \cdot \left(\lim_{n\to\infty} \frac{F_{n-1}}{F_n} \right) \left(\lim_{n\to\infty} \frac{F_{n-1}}{F_n} \right)$$

$$= L_0 \cdot \left(\lim_{n\to\infty} \frac{F_{n-1}}{F_n} \right)^2$$

Similary, we get $L_4 = L_0 \left(\lim_{n\to\infty} \frac{F_{n-1}}{F_n} \right)^3$

Generalizing these results we have

$$L_k = \left(\lim_{n\to\infty} \frac{F_{n-1}}{F_n} \right)^{k-1} \cdot L_0$$

We have the relation

$$F_n = F_{n-1} + F_{n-2}$$

$$\therefore \quad \frac{F_n}{F_{n-1}} = 1 + \frac{F_{n-2}}{F_{n-1}}$$

$$\text{or,} \quad \lim_{n\to\infty} \frac{F_n}{F_{n-1}} = 1 + \lim_{n\to\infty} \frac{F_{n-2}}{F_{n-1}}$$

$$= 1 + \lim_{n\to\infty} \frac{1}{\frac{F_{n-1}}{F_{n-2}}}$$

$$= 1 + \frac{1}{\lim\limits_{n\to\infty} \frac{F_{n-1}}{F_{n-2}}}$$

$$= 1 + \frac{1}{\lim\limits_{n\to\infty} \frac{F_n}{F_{n-1}}}$$

Let $\gamma = \lim\limits_{n\to\infty} \frac{F_n}{F_{n-1}}$

$\therefore$ We have $\gamma = 1 + \frac{1}{\gamma}$

or, $\gamma^2 = g + 1$

or, $\gamma^2 - g - 1 = 0$

or, $\gamma = \frac{-(-1) \pm \sqrt{(-1)^2 - 4.1(-1)}}{2.1}$

$= \frac{1 \pm \sqrt{5}}{2}$

Since γ is a positive real number, we have $\gamma = \frac{\sqrt{5}+1}{2} = 1{\cdot}618$, which is nothing but golden ratio or golden section.

Hence we have in general,

$$L_k = \left(\frac{1}{\gamma}\right)^{k-1} L_0 = (0{\cdot}618)^{k-1}\ L_0$$

$\therefore$ In the Golden section method the interval of uncertainty at the end of k th experiment is given by

$$L_k = (0{\cdot}618)^{k-1}\ L_0$$

7.7 Procedure of Golden Section Method

In the Fibonacci method, the location of the first two experiments are the points situated at a distance L_2^* from the two ends of the initial interval of uncertainty, where L_2^* is given by

$$L_2^* = \frac{F_{n-2}}{F_n} L_0$$

In Golden section method n is very large. Therefore L_2^* is given by

$$\begin{aligned}
L_2^* &= \lim_{n\to\infty} \frac{F_{n-2}}{F_n} L_0 \\
&= \lim_{n\to\infty}\left(\frac{F_{n-2}}{F_{n-1}}\cdot\frac{F_{n-1}}{F_n} L_0\right) \\
&= L_0 . \lim_{n\to\infty} \frac{F_{n-2}}{F_{n-1}} . \lim_{n\to\infty} \frac{F_{n-1}}{F_n} \\
&= L_0 . \lim_{n\to\infty} \frac{F_{n-1}}{F_n} . \lim_{n\to\infty} \frac{F_{n-1}}{F_n} \\
&= L_0 . \left(\lim_{n\to\infty} \frac{F_{n-1}}{F_n}\right)^2 \\
&= L_0 . \left(\frac{1}{\gamma}\right)^2 \\
&= L_0 . (\cdot 613)^2 = 0\cdot 382\ L_0 .
\end{aligned}$$

∴ In the Golden section method, the first two experiments are placed at the points x_1 and x_2 which are located at a distance $L_2^* = 0\cdot 382\ L_0$ from each end of L_0. The values of the functions f at x_1, x_2 are evaluated as $f_1 = f(x_1)$ and $f_2 = f(x_2)$. Using the assumption of unimodality, one of the two intervals $[a, x]$ and $[x_2, b]$ can be discarded. The remaining interval of uncertainty will be $L_2 = 0\cdot 618\ L_0$. The interval will contain one experiment point. The smaller distance of this experiment point from the ends of L_2 is denoted by L_3^*. The third experiment x_3 is placed in L_2 so that the current two experiments are located at a distance L_3^* from each end of L_2. Again using unimodelity we can discard one of the end intervals and the reduced internal of uncertainty at the end of 3rd experiment becomes $L_3 = (0{\cdot}618)^2 L_0$. This process is continued until the desired length of the interval of uncertainty is obtained.

7.8 Illustrative Examples

Example 7.8.1 Maximize $f(x) = \begin{cases} 2x/3, & x \le 3 \\ 5 - x, & x > 3 \end{cases}$

in the interval [1, 4] by Golden selection method up to six experiments.

Solution : We have $L_0 = 4 - 1 = 3$

Now $L_2^* = {\cdot}382\ L_0 = {\cdot}382 \times 3 = 1{\cdot}146$

The first two experiments are placed at the positions x_1 and x_2 such that

$$x_1 = 1 + L_2^* = 1 + 1{\cdot}164 = 2{\cdot}146$$

$$x_2 = 4 - L_2^* = 1{\cdot}146 = 2{\cdot}854$$

Now $f_1 = f(x_1) = \dfrac{2x_1}{3} = \dfrac{2 \times 2 \cdot 146}{3} = 1 \cdot 43066$

$$f_2 = f(x_2) = \frac{2x_2}{3} = \frac{2 \times 2 \cdot 854}{3} = 1 \cdot 90266$$

As $f_1 < f_2$ and the problem is of maximization, using unimodal property we delete the interval $[1, x_1]$. Thus the reduced interval of uncertainly is $[x, 4]$ with x_2 inside it and near to the point x_1.

The third experiment is to be placed at x_3 given by

$$4 - x_3 = x_2 - x_1$$

or, $x_3 = 4 - x_2 + x_1 = 4 - 2{\cdot}854 + 2{\cdot}146 = 3{\cdot}292$

Now, $f_3 = f(x_3) = 5 - 3{\cdot}292 = 1{\cdot}708$

Here $f_3 < f_2$. So by unimodality we delete the interval $[x_3, 4]$. The remaning interval of uncertainty becomes $[x_1, x_3]$ with x_2 inside it and near to the point x_3. The fourth experiment is placed at x_4 given by

$$x_4 - x_1 = x_3 - x_2$$

or, $x_4 = x_1 + x_3 - x_2 = 2{\cdot}146 + 3{\cdot}292 - 2{\cdot}854 = 2{\cdot}584$

Now $f_4 = f(x_4) = \frac{2x_4}{3} = \frac{2 \times 2 \cdot 584}{3} = 1 \cdot 7226$

Hence, $f_4 < f_2$. Using unimodality we delete the interval $[x_1, x_4]$. The remaning interval of uncertainty is $[x_4, x_3]$ with x_2 inside it and near to x_4.

The fifth experiment is placed at x_5 given by

$$x_3 - x_5 = x_2 - x_4$$

or, $x_5 = x_3 - x_2 + x_4 = 3{\cdot}292 - 2{\cdot}854 + 2{\cdot}584 = 3{\cdot}022$

Now $f_5 = f(x_5) = 5 - 3{\cdot}022 = 1{\cdot}978$

Since $f_5 > f_2$, using unimodal properly we delete the interval $[x_4, x_2]$. The remaining interval of uncertainty is $[x_2, x_3]$ with x_5 inside it and near to x_2.

The sixth experiment is placed at x_6 given by

$$x_3 - x_6 = x_5 - x_2$$

or, $x_6 = x_3 - x_5 + x_2 = 3{\cdot}292 - 3{\cdot}022 + 2{\cdot}854 = 3{\cdot}124$

Now $f_6 = f(x_6) = 5 - 3{\cdot}124 = 1{\cdot}876$

Since $f_6 < f_5$, using unimodality we delete the interval $[x_6, x_3]$. The final interval of uncertainty is given by $[x_2, x_6]$ is $[2{\cdot}854, 3{\cdot}124]$

7.9 Summary

The necessity of numerical methods of optimization is discussed. The inportance of one-dimensional minimization methods is solving multivariable optimization problems in described. The concept of unimodal function and its role in the elimination methods is presented. Fibonacei method and Golden section methods are discussed in detail through examples.

7.10 Self Assesment Questions

1. Minimize $f(x) = \begin{cases} 8-x, & x \leq 4 \\ x, & x \leq 4 \end{cases}$

   in the interval [1, 7] by Fibonacei method using $n = 6$

2. Minimize $f(x) = | x - 1 |$ in the interval [–1, 5] by Fibonacei method using $n = 5$.

3. Manimize $f(x) = \begin{cases} 4x/3, & x \leq 3 \\ 7-x, & x \leq 3 \end{cases}$

   in the interval [1, 5] by Golden section method upto six experiments.

4. Minimize = $f(x) = \begin{cases} 6-x, & x \leq 5 \\ 2x-9, & x \leq 5 \end{cases}$

   in the interval [2, 8] by Golden section method upto five experiments.

5. Minimize = $f(x) = \begin{cases} 2\sqrt{x}, & x \leq 1 \\ 3-x, & x \geq 1 \end{cases}$

   in the interval [0, 5] by Golden section method upto six experiments.

6. Minimize $f(x) = |x|$ in the interval [–2, 2] by Golden section method upto six experiments.

Unit 8 ☐ Unconstrained Optimization Technique

Structure

8.1 Introduction

The solution ofunconstrained optimization problem need not satisfy any constraints, Unconstrained optimization technique is important because of the following reasons

(i) Some of the most powerful and convenient methods of solving constrained optimization problems involve the transformation of the problem into one of unconstrained optimization.

(ii) The study of the unconstrained optimization methods provides the basic understanding necessary for the study of the constrained optimization methods.

Several methods are available for solving an unconstrained optimization problem. These methods are classified into two broad categories viz direct search methods and descent methods. The different methods of these two categories are shown below.

8.2 General Iterative scheme of optimization

All the unconstrained optimization methods are iterative in nature. Hence they start from an initial trial solution and proceed towards the optimum point in a sequential manner. It is importai tc ote that all the unconstrained optimization methods requires an initial point x_1 to start the iterative procedure. One method differs from another only in the method of generation the new point x_{i+1} from x_i and in testing the point x_{i+1} for optimality.

If the search d tion from x_i be s_i and the step length for movement along the search direction s_i be λ_i^* then the next point to x_i is obtained as $x_{i+1} = x_i + \lambda_i^* s_i$. Thus the terative scheme becomes.

(i) Start with an initial trial point x_1.

(ii) Find a suitable direction s_i ($i = 1$ to start with) which points is general direction of the minimum.

(iii) Find and appropriate step length λ_i^* for movement along the direction si.

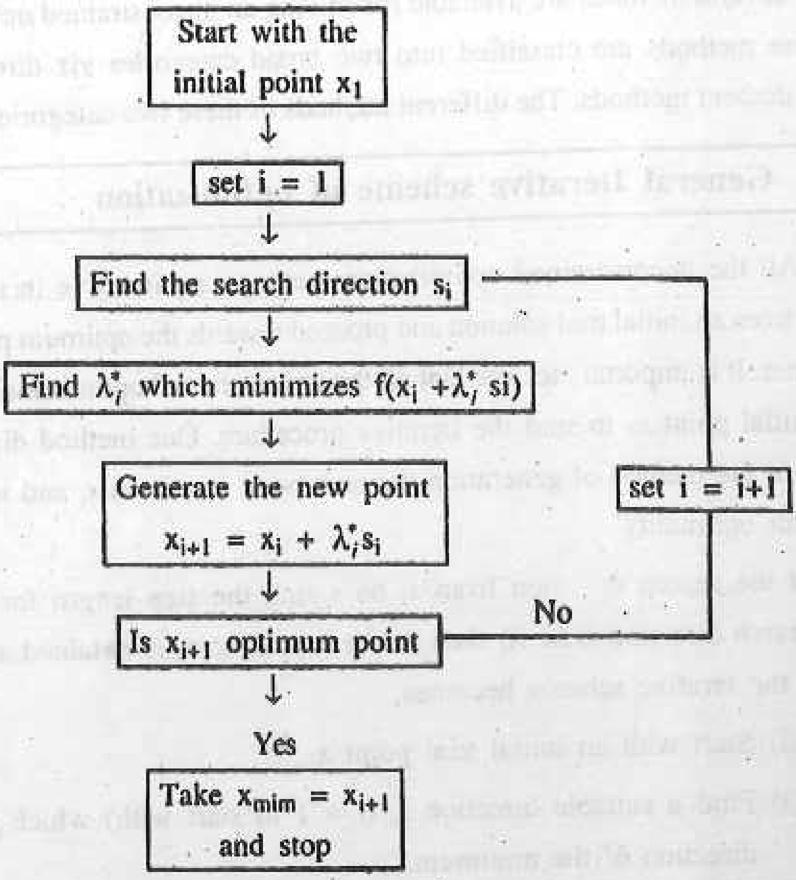
(iv) Obtain the new approximation x_{i+1} as $x_{i+1} = x_i + \lambda_i^* s_i$

(v) Test whether x_{i+1} is optimum. If x_{i+1} is optimum then stop the procedure, otherwise set new $i = i+1$ and repeat step (ii) onward.

Thus as mentioned before, the efficiency of an optimization method depends an the efficiency with which the quantities λ_i^* and si are determined to generate the new point x_{i+1} as $x_i + \lambda_i^* s_i$. To find we are to minimize $f(x_i + \lambda_i s_i)$ regarding it as a function of λ_i only.

$$f(x_i + \lambda_i^* s_i) = \min_{\lambda_i} \left(f(x_i + \lambda_i s_i) \right)$$

The flow chart for the iterative scheme may thus be shown as follows



8.3 Steepest Descent Method

In the steepest descent method of minimize a function f of n variables x_1, x_2, ..., xn we use the gradient of the the function f defined by

$$\nabla f = \left[\frac{\partial f}{\partial x_1}\,\frac{\partial f}{\partial x_2}\ldots\ldots\frac{\partial f}{\partial x_n}\right]^T$$

The gradient of f is a n-companent vector and has a very important property viz if we move along the gradient direction from any point in the n-dimensional space, then the function value increases at the fastest rate. To prove this properly we first define directional derivalive.

Definition 8.3.1 Directional Devivative : The directional devivative of f(x) in the direction of the unit vector y is defined as the following limit

$$\lim_{t \to 0} \frac{f(x+ty) - f(x)}{t}$$

The directional derivative of f(x) in the direction y is thus given by using Taylor's theorem

$$\lim_{t \to 0} \frac{\{f(x) + (ty)\nabla f(x) + \text{terms of higher degree in t}\} - f(x)}{t}$$

$= y'\nabla f(x)$

∴ The directional derivative of f(x) in the direction of unit vector y

$= y'\nabla f(x)$

= rate of change of f(x) in the direction of y.

Theorem 8.3.1 Prove that f(x) increases at the fastest rate in the direction of ∇f.

Proof : We have that the rate of change of f(x) in the direction of the unit vector y is $y'\nabla f(x)$ .... (1)

Now the unit vector in the direction of the gradient vector ∇f is $\nabla f / |\nabla f|$. Therefore, the rate of change of $f(x)$ in the direction of the gradient vector

$$= \left(\frac{\nabla f}{|\nabla f|}\right)' \nabla f = \frac{(\nabla f)'(\nabla f)}{|\nabla f|} = \frac{|\nabla f|^2}{|\nabla f|} = |\nabla f| \text{ ....(2)}$$

Since $|\nabla f| > 0$, it follows the $f(x)$ increases in the direction of ∇f.

Using cauchy schwarz inequality we have

$$|y'\nabla f| \le |y|\,|\nabla f| = |\nabla f| \;[\because |y| = 1] \text{.......... (3)}$$

From (1), (2) and (3) it follows that the rate of change of $f(x)$ in the direction of ∇f is greater than that in the direction of any unit vector y. In other words $f(x)$ increases at the fastest rate in the direction of ∇f.

Note : Since $f(x)$ increases at the fastest rate in the direction of ∇f, it follows that $f(x)$ decreases at the fastest rate in the direction of $-\nabla f$. Thus the direction of ∇f and $-\nabla f$ are respectively the directions of the steepest ascent and steepest descent.

8.4 Iterative Scheme of Steepest Descent Method

The steepest descent method uses the properly that a function $f(x)$ decreases a the fastest rate in the direction of $-\nabla f$. Thus at x_i the function decreases at the fastes rate along the direction si given by $s_i = [-\nabla f]_{x_i} = -\nabla f_i$.

The iterative scheme of steepest descent method is given below.

(i) Start with an initial point x_1.

(ii) Take the search direction s_i at x_i (i = 1 to start with) as $s_i = [-\nabla f]_{x_i}$ and denote it by $-\nabla f_i$.

(iii) Find the step length λ_i^* for movement along si which minimizes $f(x_i + \lambda_i^* s_i$

(iv) Obtain the new appromimation point x_{i+1} as $x_{i+1} = x_i + \lambda_i^* s_i$

(v) Test whether x_{i+1} is optimum. If x_{i+1} is optimum then stop the procedure Otherwise set new $i = i+1$ and repeat setp (ii) onward.

8.5 Illustrative Examples

Example 8.5.1 Using steepest descent method minimize f = $x_1^2 + x_2^2 + 2gx_1 + 2fy_1 + c$ starting from the point

Solution : Here $f(x_1, x_2) = x_1^2 + x_2^2 + 2gx_1 + 2fy_1 + c$

$\therefore$ The gradient of f is given by

$$\nabla f = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 + 2g \\ 2x_2 + 2f \end{bmatrix}$$

The starting point is $x_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. Using steepest desant method the search directio at x_1 is given by

$$s_i = [-\nabla f]_{x_i} = \begin{bmatrix} -2\alpha - 2g \\ -2\beta - 2f \end{bmatrix}$$

The step length λ_1^* is obtained by minimising $f(x_1 + \lambda_1 s_1)$ with respect to λ_1.

$$\text{Now } (x_1 + \lambda_1 s_1) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \lambda_1 \begin{bmatrix} -2\alpha - 2g \\ -2\beta - 2f \end{bmatrix} = \begin{bmatrix} \alpha - 2\lambda_1\alpha - 2\lambda_1 g \\ \beta - 2\lambda_1\beta - 2\lambda_1 f \end{bmatrix} = \begin{bmatrix} \gamma \\ \delta \end{bmatrix}.$$

Where $\gamma = \alpha + \lambda_1 (-2\alpha - 2g)$

and $\delta = \beta + \lambda_1 (-2\beta - 2f)$

$\therefore\ f(x_1 + \lambda_1 s_1) = \gamma^2 + \delta^2 + 2g\gamma + 2f\delta + c$

For minimum value of f we have $\frac{df}{d\lambda_1} = 0$. This gives $\frac{\partial f}{\partial \gamma}\frac{\partial \lambda}{\partial \lambda_1} + \frac{\partial f}{\partial \delta}\cdot\frac{\partial \delta}{\partial \lambda_1} = 0.$

or, $(2\gamma + 2g)(-2\alpha - 2g) + (2\delta + 2f)(-2\beta - 2f) = 0$

or, $(\gamma + g)(\alpha + g) + (\delta + f)(\beta + f) = 0$

or, $\{\alpha + \lambda_1(-2\alpha - 2g)\}(\alpha + g) + \{\beta + \lambda_1(-2\beta - 2f) + f\}(\beta - f) = 0$

or, $(\alpha + g)^2 - 2\lambda_1(\alpha + g)^2 + (\beta + f)^2 - 2\lambda_1(\beta + f)^2 = 0$

or, $(1 - 2\lambda_1)[(\alpha + g)^2 + (\beta + f)^2] = 0$

or, $1 - 2\lambda_1 = 0$

or, $\lambda_1 = \frac{1}{2}$

$\therefore\ \lambda_1^* = \frac{1}{2}$

Now, x_1 is given by $x_2 = x_1 + \lambda_1^* s_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -2\alpha - 2g \\ -2\beta - 2f \end{bmatrix} = \begin{bmatrix} \alpha - \alpha - g \\ \beta - \beta - f \end{bmatrix} = \begin{bmatrix} -g \\ -f \end{bmatrix}$

The gradient of f at x_2 is given by

$$[\nabla f]_{x_2} = \begin{bmatrix} 2(-g) + 2g \\ 2(-f) + 2f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This shows that x_2 is the optimum point

$$\therefore \; x_{opt} = x_2 = \begin{bmatrix} -g \\ -f \end{bmatrix}$$

Example 8.5.2 Using steepest descent method minimize $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$ starting from the point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Solution : Here $f = f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$ and the starting point is $x_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The gradient of f is given by

$$\nabla f = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{bmatrix}$$

$$\therefore \; \nabla f_1 = [\nabla f]_{x_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The search direction at x_1 is given by $s_1 = -\nabla f_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

To find x_2 we are to find the optimal step length λ_1^*. For this we are to minimize $f(x_1 + \lambda_1 s_1)$ with respect to λ_1.

Now $x_1 + \lambda_1 s_1 = -\lambda_1 - \lambda_1 + 2\lambda_1^2 - 2\lambda_1^2 + \lambda_1^2 = \lambda_1^2 - 2\lambda_1$

For minimum value of f we have $\dfrac{df}{d\lambda_1} = 0$.

From this we have $2\lambda_1 - 2 = 0$

or, $\lambda_1 = 1$

$\therefore \; \lambda_1^* = 1$

Thus we obtain x_2

$$x_2 = x_1 + \lambda_1^* s_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The gradient of f at x_2 is given by

$$\nabla f_2 = [\nabla f]_{x2} = \begin{bmatrix} 1-4+2 \\ -1-1+2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore$ x_2 is not an optimum point. So we proceed to the next iteration.

The search direction at x_2 is given by

$$s_2 = -\nabla f_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

To find x_3 we find the step length λ_3^* by minimizing $f(x_2 + \lambda_2 s_2)$ with respect to λ_2.

$$\text{Now } x_2 + \lambda_2 s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+\lambda_2 \\ 1+\lambda_2 \end{bmatrix}$$

$$\therefore\ f(x_2 + \lambda_2 s_2) = (-1 + \lambda_2)(1 + \lambda_2) + 2(-1 + \lambda_2)^2 + 2(-1 + \lambda_2)(1 + \lambda_2) + (1 + \lambda_2)^2$$

$$= -1 + \lambda_2 - 1 - \lambda_2 + 2 - 4\lambda_2 + 2\lambda_2^2 - 2 + 2\lambda_2^2 + 1 + 2\lambda_2 + \lambda_2^2$$

$$= -1 - 2\lambda_2 + 5\lambda_2^2$$

To minimize f we set $\frac{df}{d\lambda_2} = 0$

Form this we have $-2 + 10\lambda 2 = 0$

or, $\lambda_2 = +\frac{1}{5}$

$\therefore\ \lambda_2^* = \frac{1}{5}$

$$\text{Hence } x_3 = x_2 + \lambda_2^* s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0\cdot 8 \\ 1\cdot 2 \end{bmatrix}$$

The gradient of f at x_3 is given by

$$\nabla f_3 = [\nabla f]_{x_2} = \begin{bmatrix} 1+4(-0\cdot 8)+2(1\cdot 2) \\ -1+2(-0\cdot 8)+2(1\cdot 2) \end{bmatrix} = \begin{bmatrix} 0\cdot 2 \\ -0\cdot 2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ x_3 is not optimum and we proceed to the next iteration.

The search direction at x_3 is given by

$$s_3 = -\nabla f_3 = \begin{bmatrix} -0\cdot 2 \\ 0\cdot 2 \end{bmatrix}$$

To find x_4 we are to find the step length λ_4^* by minimizing $f(x_3 + \lambda_3 s_3)$ with respect to λ_3.

$$\text{Now } x_3 + \lambda_3 s_3 + \begin{bmatrix} -0\cdot 8 \\ 1\cdot 2 \end{bmatrix} + \lambda_3 \begin{bmatrix} -0\cdot 2 \\ 0\cdot 2 \end{bmatrix} = \begin{bmatrix} -0\cdot 8-\lambda_3 0\cdot 2 \\ 1\cdot 2+\lambda_3 0\cdot 2 \end{bmatrix}$$

$$\therefore\ f(x_3 + \lambda_3 s_3) = (-0{\cdot}8 - 0{\cdot}2\,\lambda_3) - (1{\cdot}2 + 0{\cdot}2\,\lambda_3) + 2\,(-0{\cdot}8 - 0{\cdot}2\lambda_3)^2$$
$$+ 2\,(-\,0{\cdot}8 - 0{\cdot}2\,\lambda_3)\,(1{\cdot}2 + 0{\cdot}2\,\lambda_3) + (1{\cdot}2 + 0{\cdot}2\,\lambda_3)^2$$

$$= 0\cdot 04\lambda_3^2 - 0\cdot 08\lambda_3 - 1\cdot 20$$

To minimize f we set $\dfrac{df}{d\lambda_3} = 0$

To gives $2 \times 0{\cdot}04\lambda_3 - 0{\cdot}08 = 0$

or, $\lambda_3 = 1$

$\therefore\ \lambda_3^* = 1$

$$\text{Hence } x_4 = x_3 + \lambda_3^* s_3 = \begin{bmatrix} -0\cdot 8 \\ 1\cdot 2 \end{bmatrix} + 1 \begin{bmatrix} -0\cdot 2 \\ 0\cdot 2 \end{bmatrix} = \begin{bmatrix} -1\cdot 0 \\ 1\cdot 4 \end{bmatrix}$$

The gradient of f at x_4 is given by

$$\nabla f_4 = [\nabla f]_{x_4} = \begin{bmatrix} 1+4(-1\cdot 0)+2(1\cdot 4) \\ -1+2(-1\cdot 0)+2(1\cdot 4) \end{bmatrix} = \begin{bmatrix} -0\cdot 20 \\ -0\cdot 20 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So x_4 is also not optinum and we are to continue the iterations until we have

$\nabla f_n \simeq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and then x_n is taken has the optimum point.

Convergence Creteria: The following criteria can be used to terminate the iteratice process.

(i) $\left| \frac{f(x_{i+1}) - f(x_i)}{f(x_i)} \right| \leq \in$

(ii) $\left| \frac{\partial f}{\partial x_i} \right| < \in$ for all $i = 1, 2, ..., n$

(iii) $| x_{i+1} - x_i | \leq \in$

8.6 Quadratically Convergent Method

Example 8.6.1 A minimization method is called quadratically convergent method if it locates the minimum of general function in no more than a pre-determined number of operations and if the limitting number of operations is directly related to the number of variates.

Definition 8.6.2 Let A be an nxn pymmetric matrix. A set of n vectors s_1, s_2, ..., sn is said to be A conjugate directions if $s_i^T A\ s_i = 0$ for all i ≠ j, i, j = 1,2, 3, ..., n.

Example 8.6.1 Find the conjugate direction for the symmetric matrix $\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$

Solution : Let A = $\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$ and A-conjugate direction be $s_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ and s_2 $= \begin{bmatrix} \gamma \\ \delta \end{bmatrix}$

$\therefore\ \ s_i^T A\ s_2 = 0$

or, $[\alpha\ \beta]\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} \gamma \\ \delta \end{bmatrix} = 0$

or, $\gamma\,(2\alpha - 3\beta) + \delta\,(-3\alpha + 2\beta) = 0$

Let, $\alpha = 1, \beta = 2, \gamma = 1$ $\quad \therefore\ -1\,(2.1 - 3.2) + \delta\,(-3.1 + 2.2) = 0$

or, $4 + \delta\,(+1) = 0$

or, $\delta = -4$

Thus the conjugate direction are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -4 \end{bmatrix}$

We note that for a given matrix there are many conjugate directions.

Matrix representation of quadratic expression :

Any quadratic expression can be expressed with the help of matrices as

$$\frac{1}{2}x^T Ax + B^T x + c$$

Where A is asymmetric matrix

eg. $3x_1^2 + 2x_2^2 + 4x_3^2 + 4x_1x_2 - x_2x_3 + 3x_3x_1 + 3x_1 - 2x + x_3 + 7$

can be written as $\frac{1}{2}x^T Ax + B^T x + c$

Where $A = \begin{bmatrix} 6 & 4 & 3 \\ 4 & 4 & -1 \\ 3 & -1 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$, $C - 7$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

We state the following important theorem.

Theorem 8.6.1 If quadratic function $Q(x) = \frac{1}{2}x^T Ax + B^T x + c$ is minimized sequentially once along each direction of a set of n A-conjugate directions them the global minimum of Q(x) will be located at a before the n th setp regareless of the starting point and the order in which the directions are used.

8.7 Newton's Method

If the function $f(x)$ is continuously differentiable then the local minimum point x^* is given by $[\nabla f]_{x^*} = 0$. Solving the set of n nonlinear equations $\nabla f = 0$ we get the optimal point x^*.

Newton's method : To get the minimum point x^* of the continuously differentiable function $f(x)$ we are to solve the n nontinear equation $\nabla f = 0$. To solve these n nonlinear equations by the Newton's method, we first linearize the set of equation about the i th appromimations x_i to the minimum point x^* of f.

Let $x^* = x_i + s$ and $\nabla f = g$

From $[\nabla f]_{x^*} = 0$ we have $g(x^*) = 0$ or, $g(x_i + s) = 0$

By Taylor's series expansion we get

$g(x_i) + [J]_{x_i} s + \ldots = 0$ where $[J]_{x_i}$ is the matrix of second partial devivatives of f evaluated at the point Negleeting the higher order terms we get

$$g(x_i) + [J]_{x_i} s = 0$$

or, $g_i + J_i s = 0$ where $g(x_i) = gi$ and $[J]_{x_i} = J_i$. If J_i is non singular, then we have

$$S = -J_i^{-1} g_i$$

But the higher order terms are not negligible in general. Hence an iterative procedure has to be used to find the improved approximations. The iterative scheme is given by

$$x_{i+1} = x_i + s_i = x_i - J_i^{-1} g_i$$

If J is nonsingular then it can be shown that the sequence of points $x_1, x_2, \ldots, x_i, \ldots$ converges to the actual solution x^* from any initial point x_1 sufficiently close to the solution x^*.

Theorem 8.7.1 If $f(x)$ is a quadratics then the minimum point can be obtained in a single step by Newton's method.

Proof : Let $f(x) = \frac{1}{2}x^T Ax + B^T x + c$ & the minimum point be x^*. Then $[\nabla f]_{x^*} = 0$

or, $[Ax + B]_{x^*} = 0$

or, $Ax^* + B = 0$

or, $x^* = -A^{-1}B$.

From $f(x) = \frac{1}{2}x^T Ax + B^T x + c$ we have $\nabla f = Ax + B$ and J = matrix of second partial derivatives of f = A. By Newton's method we have

$$\begin{aligned}
x_{i+1} &= x_i - J_i^{-1} g_i \\
&= x_i - A^{-1}(Ax_i + B) \\
&= x_i - A^{-1}Ax_i + A^{-1}B) \\
&= x_i - x_i - A^{-1}B \\
&= -A^{-1}B = x^*
\end{aligned}$$

$\therefore\ x_2 = -A^{-1}B = x^*$ for any starting point x_1.

Thus the answer is obtained in a single step.

Example 8.7.1 Using Newton's method

minimize $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + 2x_2^2$ with $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as starting point.

Solution : Here $f = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + 2x_2^2$

$$\therefore\ \frac{\partial f}{\partial x_1} = 1 + 4x_1 + 2x_2, \quad \frac{\partial f}{\partial x_2} = -1 + 2x_1 + 2x_2$$

$$\frac{\partial^2 f}{\partial x_1^2} = 4, \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = 2, \quad \frac{\partial^2 f}{\partial x_2^2} = 2$$

The starting point is $x_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\nabla f_1 = [\nabla f]_{x_1} = \begin{bmatrix} 1+0+0 \\ -1+0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\& \quad J1 = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\therefore \quad J_1^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

We have $x_1 = x_2 - J_1^{-1}\nabla f_1$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}+\frac{1}{2} \\ -\frac{1}{2}-1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3/2 \end{bmatrix}$$

Now $\nabla f_2 = [\nabla f]_{x_2} = \begin{bmatrix} 1+4(-1)+2(3/2) \\ -1+2(-1)+2(3/2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

As $\nabla f_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} -1 \\ 3/2 \end{bmatrix}$ is the optimum point.

8.8 Davidon-Fletcher-Powell Method (Variable Metric Method)

Davidon-Fletcher-Powell method is an important quasi-Newton method. This method is the best general purpose unconstrained optimization technique making use of the derivatioes.

The iterative procedure of this method is as follows :

(i) Start with an initial point x_1 and a nxn positive definite symmetric matrix H_1. Usually H_1 is taken as the identely matrix I. Set iteratio number is i = 1.

(ii) Compute the gradient of the function f at the point x_i i.e., compute $\nabla f_i = [\nabla f]_{x_i}$

Take $s_i = H_i \nabla f_i$ as the search direction at x_i.

(iii) Find the optimal step length λ_i^* in the direction s_i and set $x_{i+1} = x_i + \lambda_i^* s_i$

(iv) Test the new point x_{i+1} for optimality. If x_{i+1} is optimal, terminate the iterative process. Otherwise go to setp (v).

(v) Update H_i to H_{i+1} as

$$H_{i+1} = H_i + M_i + N_i$$

Where $M_i = (\lambda_i^* s_i s_i^T) / (s_i^T Q_i)$

$$Ni = -(H_i Q_i)(H_i Q_i)^T / (Q_i^T H_i Q_i)$$

$$Q_i = \nabla f_{i+1} - \nabla f_i$$

(vi) Set the new iteration number i = i + 1 and go to step (ii).

8.9 Illustrative Examples

Example 8.9.1 Using Davidon Fletcher-Powell method minimize $f(x_1, x_2) = 2x_1^2 + 4x_2^2 - 12x_1 + 16x_2 + 41$ with $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as starting point.

Solution : Here $f = 2x_1^2 + 4x_2^2 - 12x_1 + 16x_2 + 41$

$$\therefore \quad \nabla f = \begin{bmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 - 12 \\ 8x_2 + 16 \end{bmatrix}$$

Thus $\nabla f_1 = [\nabla f]_{x_1} = \begin{bmatrix} 4 - 12 \\ 8 + 16 \end{bmatrix} = \begin{bmatrix} -8 \\ 24 \end{bmatrix}$

We take $H_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \quad s_1 = -H_1 \nabla f_1 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -8 \\ 24 \end{bmatrix} = \begin{bmatrix} -8 \\ 24 \end{bmatrix}$$

To find the minimizing step length λ_1^* along s_1, we minimize

$$f(x_1 + \lambda_1 s_1) = f(1 + 8\lambda_1,\ 1 - 24\lambda_1)$$

$$= 2\,(1 + 8\lambda_1)^2 + 4\,(1 - 24\lambda_1)^2 - 12\,(1 + 8\lambda_1) + 16\,(1 - 24\lambda_1) + 41$$

$$= 2 + 32\lambda_1 + 128\lambda_1^2 + 4 - 192\lambda_1 + 2304\lambda_1^2 - 12 - 96\lambda_1 + 16 - 384\lambda_1 + 41$$

$$= 2432\lambda_1^2 - 640\lambda_1 + 51$$

We set $\dfrac{df}{d\lambda_1} = 0$

$$\therefore \quad 2432 \times 2\lambda 1 - 640 = 0$$

$$\text{or,}\quad \lambda 1 = \frac{640}{2 \times 2432} = \frac{10}{76} = 0{\cdot}1316$$

$$\therefore \quad \lambda_1^* = 0{\cdot}1316$$

$\therefore$ The second approximation is given by

$$x_2 = x_1 + \lambda_1^* s_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \cdot 1316 \begin{bmatrix} 8 \\ -24 \end{bmatrix} = \begin{bmatrix} 2 \cdot 0528 \\ -2 \cdot 1584 \end{bmatrix}$$

$$\text{Now}\quad \nabla f_2 = [\nabla f]_{x_2} = \begin{bmatrix} 4 \times 2 \cdot 0528 - 12 \\ 8 \times (-2 \cdot 1584) + 16 \end{bmatrix} = \begin{bmatrix} -3 \cdot 7888 \\ -1 \cdot 2672 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ x_2 is not optimum point

To update the matrix H_1 we compute

$$Q_1 = \nabla f_2 - \nabla f_1 = \begin{bmatrix} -3 \cdot 7888 \\ -1 \cdot 2672 \end{bmatrix} - \begin{bmatrix} -8 \\ 24 \end{bmatrix} = \begin{bmatrix} 4 \cdot 2112 \\ -25 \cdot 2672 \end{bmatrix}$$

$$\therefore \quad S_1^T Q_1 = [8 - 24]\begin{bmatrix} 4\cdot 2112 \\ -25\cdot 2672 \end{bmatrix} = 640\cdot 1024$$

$$S_1 S_1^T = \begin{bmatrix} 8 \\ -24 \end{bmatrix}[8 \quad -24] = \begin{bmatrix} 64 & -192 \\ -192 & 576 \end{bmatrix}$$

$$H_1 Q_1 = Q_1 = \begin{bmatrix} 4\cdot 2112 \\ -25\cdot 2672 \end{bmatrix}$$

$$\therefore \quad (H_1Q_1)(H_1Q_1)^T = \begin{bmatrix} 4\cdot 2112 \\ -25\cdot 2672 \end{bmatrix} = [\, 4{\cdot}2112 \; - 25{\cdot}2672 \,]$$

$$= \begin{bmatrix} 17\cdot 7242 & -106\cdot 4052 \\ -106\cdot 4052 & 638\cdot 4314 \end{bmatrix}$$

Also $Q_1^T(H_1Q_1) = [4\cdot 2112 \quad -25\cdot 2672]\begin{bmatrix} 4\cdot 2112 \\ -25\cdot 2672 \end{bmatrix} = 656\cdot 1656$

$$\therefore \quad N_1 = -\frac{(H_1Q_1)(H_1Q_1)^T}{Q_1^T(H_1Q_1)} = -\frac{1}{656\cdot 1656}\begin{bmatrix} 17\cdot 7242 & -106\cdot 4052 \\ -106\cdot 4052 & 638\cdot 4314 \end{bmatrix}$$

$$= -\begin{bmatrix} 0\cdot 027 & -0\cdot 1625 \\ -0\cdot 1625 & 0\cdot 973 \end{bmatrix}$$

$$M_1 = \frac{\lambda_1^* S_1 S_1^T}{S_1^T Q_1} = \frac{0\cdot 1316}{640\cdot 1024}\begin{bmatrix} 64 & -192 \\ -192 & 576 \end{bmatrix} = \begin{bmatrix} 0\cdot 0132 & -0\cdot 0395 \\ -0\cdot 0395 & 0\cdot 1184 \end{bmatrix}$$

$$\therefore \quad H_2 + H_1 + M_1 + N_1$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0\cdot 0132 & -0\cdot 0395 \\ -0\cdot 0395 & 0\cdot 1184 \end{bmatrix} + \begin{bmatrix} -0\cdot 027 & 0\cdot 1625 \\ 0\cdot 1625 & -0\cdot 973 \end{bmatrix}$$

$$= \begin{bmatrix} 0\cdot 8062 & 0\cdot 123 \\ 0\cdot 123 & 0\cdot 1454 \end{bmatrix}$$

Hence $S_2 = -H_2 \nabla f_2 = -\begin{bmatrix} 0\cdot 8062 & 0\cdot 123 \\ 0\cdot 123 & 0\cdot 1454 \end{bmatrix}\begin{bmatrix} -3\cdot 7888 \\ -1\cdot 2672 \end{bmatrix} = \begin{bmatrix} 3\cdot 21 \\ 0\cdot 622 \end{bmatrix}$

To find the minimizing step length along S_2 we are to minimize $f(x_2 + \lambda_2 S_2)$ $= f(3{\cdot}21\lambda_2, -0{\cdot}9472, 0{\cdot}622\lambda_2 - 0{\cdot}1584) = 2(3{\cdot}21\lambda_2 - 0{\cdot}9472)^2 + 4(0{\cdot}622\lambda_2 - 0{\cdot}1584)$ $- 12\,(3{\cdot}21\lambda_2 - 0{\cdot}9474) + 16\,(0{\cdot}622\lambda_2 - 0{\cdot}1584) + 41$

We set $\dfrac{df}{d\lambda_2} = 0$

This gives $\lambda 2 = 0{\cdot}292$

$\therefore\ \lambda_2^* = 0{\cdot}292$

The third approximation is given by

$$x_3 = x_2 + \lambda_2^* s_2 = \begin{bmatrix} 2\cdot 0528 \\ -2\cdot 1584 \end{bmatrix} + 0\cdot 292 \begin{bmatrix} 3\cdot 21 \\ 0\cdot 622 \end{bmatrix} = \begin{bmatrix} 2\cdot 99 \\ -1\cdot 98 \end{bmatrix}$$

Now, $\nabla f_3 = [\nabla f]_{x_3} = \begin{bmatrix} 4\times 2\cdot 99 - 12 \\ 8\times(-1\cdot 98)+16 \end{bmatrix} = \begin{bmatrix} -0\cdot 04 \\ 0\cdot 16 \end{bmatrix} \simeq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore\ x_3 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2\cdot 99 \\ -1\cdot 98 \end{bmatrix}$ *i.e.*, $x_1 = 2{\cdot}99$, $x_2 = -1{\cdot}98$ is the optimum point.

8.10 Summary

The unit is devoted to some unconstrained method of optimization viz. steepest descent method, Quadralically convergent method, Newton's method and Dairlon-Fletches-Powell method. These methods are explained with examples.

8.11 Self Assessment Questions

1. Using steepest descent method minimize the function $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 4x_1 + 3x_3 + 9$ starting from the point (1, 2, 30).

2. Using steepest descent method minimize $f(x_1, x_2) = 2x_1 - x_2 + 8x_1^2 + 4x_1x_2 + x_2^2$ starting from the point (0, 0).

3. Find the conjugate directions for the matrix $\begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}$

4. Using Davidon Fletcher and Powell method minimize $f(x_1, x_2) = x_1 - 2x_2 + 2x_1^2 + 4x_1x_2 + 4x_2^2$ starting from the point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

5. Using Davidon-Fletcher Powell method minimize $f(x_1, x_2) = 8x_1^2 + 4x_2^2 - 24x_1 + 16x_2 + 35$ with $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ as the starting point.

6. Using Davidon-Fletcher Powell method minimize $f(x_1, x_2) = 2x_1 + 3x_2 + 8x_1^2 + 12x_1x_2 + 9x_2^2$ with $\begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix}$ as the starting point.

Unit 9 ❑ Constrained Optimization Techniques

Structure

9.1 Introduction

The constrained optimization problem is

Minimize $f(x)$

subject to $g_j, (x) \leq 0, \; j = 1, 2, \ldots, m$

There are many techniques to solve a constrained non linear programming problem. All these methods canbe classified as follows.

Constrained optimization techniques

Direct methods

(i) Heuristic search methods

(ii) Methods of feasible directions

- (a) Zoutendijlis method
- (b) Gradient projection method

(iii) Cutting plane

Indirect methods

(i) By the transformation of variables

(ii) Penalty function methods

- (a) Interior penalty function methods
- (b) Exterior penalty function methods

In the direct methods, the constraints are handled in an explicit manner whereas in most of the indirect methods, the constrained problem is solved as a sequence of unconstrained minimization problems.

In this unit we discuss only cutting plane method.

9.2 Cutting Plane Method

In the cutting plane method, the nonlinear constraints are linearized by using Taylor's series expansion thereby approximating the feasible region by linearized envelopes. Assuming that the objective function is linear, we can solve the approximating LPP by this simplex method. If the solution of the LPP is not sufficiently accurate, we relinearize the binding constraints about the current point and formulate a new approximating LPP as solve it using the simplex method. We repeat this procedure until asufficiently accurate solution is found. We note that the approximating linear constraint cut off a portion of the existing feasible region. Hence the method is called cutting plane method.

To apply cutting plane method it is necessary that the objective function is linear. If the objective function is non-linear then we can formulate an equivalent optimization problem with linear objective function as follows.

Let the given problem be

Find $(x_1, x_2, \ldots\ldots, x_n)$ which minimize $f(x_1, x_2, \ldots\ldots, x_n)$

subject to the constraints gj $(x_1, x_2, \ldots\ldots, x_n) \leq 0$, j = 1, 2, ..., m.

We introduced a new variable x_{n+1} and transform this problem into an equivalent problem as follows

Find $(x_1, x_2, \ldots\ldots, x_n, x_{n+1})$ which minimize $0x_1, 0x_2, + \ldots + 0x_n, + x_{n+1}$ subject to the cosntraints gj $(x_1, x_2, \ldots\ldots, x_n) \leq 0$ j = 1, 2, ..., m and $g_{m+1}(x_1, x_2, \ldots\ldots, x_{n+1}) = f(x_1, x_2, \ldots\ldots, x_n) - x_{n+1} \leq 0$

Thus, without loss of generally, we can assume that the given problem is

Minimize $f(x) = f(x_1, x_2, \ldots\ldots, x_n) = \bar{c}^T x = c_1x_1 + c_2x_2 + \ldots\ldots + c_{nxn}$

subject to the constraints $g_j(x) = g_j(x_1, x_2, \ldots, x_n) \leq 0$ j = 1, 2, ..., m

The iterative procedure of cutting plane method can be stated as follows :

9.3 Algorithm of Cutting Plane Method

(i) Start with an initial point x_1 and set the iteration number as i = 1. The point x_1 need not be feasible

(ii) Linearize the nonlinear constraint functions $g_j(x)$ about the point x_i as

$g_j(x) \simeq g_j(x_j) + [\nabla g_j(x_i)]^T (x - x_i), \; j = 1, 2, \ldots\ldots, m$

(iii) Formulate the approximating linear programming problem as

Minimize $f(x) = c^T x$

subject to $g_j(x_i) + [\nabla g_j(x_j)]^T (x - x_i) \leq 0, \; j = 1, 2, \ldots\ldots, m$

(iv) Solve the approximating LPP to obtain the solution vector x_{i+1}.

(v) Evaluate the original constraints at x_{i+1} i.e., find gj (x_{i+1}) for all j = 1,2, ..., m.

(vi) If $g_j(x_{i+1}) \leq \in$ for all j = 1, 2, ..., m where $\in$ is a prescribed small positive tolerance then all the original constaints can be assumed to have been satisfied.

Hence stop the procedure and take $x_{opt} = x_{i+1}$

It $g_i(x_{i+1}) >$ for some value of j, find the most violated constranit as

$g_k(x_{i+1}) = \max [g_j(x_{i+1})]$

Relinearize the constrant $g_k(x)$ 0 about the point x_{i+1} as

$g_k(x) \simeq g_k(x_{i+1}) + [g_k(x_{i+1})]^T (x - x_{i+1}) \leq 0$

and add this linear constraint to the previous approximating LPP.

(vii) Set the new iteration number i = i+1 and increase the total number of constraints in thenew approximationg LPP by one and go to step (iv).

Note : To avoid the unbounded solution of the first approximating LPP we may take the first approximating LPP as

Minimize $f(x)$ $c^T x$

subject to $l_i\ x_i\ ll_i$, i = 1, 2, ..., n

Where l_i and ll_i are chosen as lower and upper bounds of x_i take theoptimum solution of this first approximating LPP as x_1 in this first step.

9.4 Illustrative Examples

Example 9.4.1 Using cutting plane method

Maximize $f(x_1, x_2) = 7 - 2x_1 - 4x_2$

subject to $(x_1 - 4)^2 + 2(x_2 - 3)^2 \leq 12$ taking $\in = 0{\cdot}03$

$x_1 + 2x_2 \leq 6$

$1 \leq x_1 \leq 6$

$1 \leq x_2 \leq 6$

Solution : We first consider the LPP

Maximize $f(x_1, x_2) = 7 - 2x_1 - 4x_2$

subject to $x_1 + 2x_2 \leq 6$

$1 \leq x_1 \leq 6$

$1 \leq x_2 \leq 6$

The extreme point of the feasible region are A (1,1), B (4, 1) and C (1, 5/2).

The value of the objective functions are

(1, 1) = 1, (4, 1) = – 5, (1, $^5/_2$) = – 5

∴ The optimal solution of the LPP is (1, 1)

∴ The first approximationg point is $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Let $g(x_1, x_2) = (x_1 - 4)^2 + 2(x_2 - 3)^2 - 12$

∴ The given non-linear constraint is $g(x_1, x_2) \leq 0$

We gave $g(x) = \begin{bmatrix} 2(x_1 - 4) \\ 4(x_2 - 3) \end{bmatrix}$

Now $g(x_1) = g(1, 1)\ (1 - 4)^2 + 2(1 - 3)^2 - 12 = 5 > \in = 0{\cdot}03$.

Hence we linearize g (x) about x_1 as follows to replace

$$g(x)\ 0 \text{ as } g(x_1) + [\nabla g(x_1)]^T \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \le 0$$

$$\text{or, } 5 + [-6, -8] \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \le 0$$

or, $5 + (-6)(x_1 - 1) + (-8)(x_2 - 1) \le 0$

or, $-6x_1 - 8x_2 + 19 \le 0$

or, $6x_1 + 8x_2 \ge 19$

We now consider the following LPP by adding the constraint $6x_1 + 8x_2 \ge 19$ as

Maximize $f = 7 - 2x_1 - 4x_2$

subject to $x_1 + 2x_2 \le 6$

$6x_1 + 8x_2 \ge 19$

$1 \le x_1 \le 6$

$1 \le x_2 \le 6$

The extreme points of the feasible region are

A_1 $(1, {}^{13}/_8)$, A_2 $({}^{11}/_6, 1)$, B $(4, 1)$ and C $(1, {}^{5}/_2)$

The values of the objective function are

$f(1, {}^{13}/_8) = -{}^{3}/_2, f({}^{11}/_6, 1) = -{}^{2}/_3, f(4, 1) = -5, f(1, {}^{5}/_2) = -5$

∴ The optimal solution of the LPP is

$x_1 = {}^{11}/_6, x_2 = 1$

∴ We take the next appromimality point as $x_2 = \begin{bmatrix} 11/6 \\ 1 \end{bmatrix}$.

Now $g(x_2) = g({}^{11}/_6, 1) = \left(\frac{11}{6} - 4\right)^2 + 2(1 - 3)^2 - 12 = \frac{25}{36} = 0{\cdot}69 > \epsilon = 0{\cdot}03$

We relinearize $g(x)$ about x_2 as follows and consider

$$g(x) \le 0 \text{ as } g(x_2) + [\nabla g(x_2)]^T \begin{bmatrix} x_1 - 11/6 \\ x_2 - 1 \end{bmatrix} \le 0$$

or, $$\frac{25}{36} + \left[-\frac{13}{3} \quad -8\right] \begin{bmatrix} x_1 - 11/6 \\ x_2 - 1 \end{bmatrix} \le 0$$

or, $165x_1 + 288x_2 \ge 599$

We add this constraint to the previous LPP to get the following LPP

Maximize $f = 7 - 2x_1 - 4x_2$

subject to $x_1 + 2x_2 \le 6$

$6x_1 + 8x_2 \ge 19$

$156x_1 + 288x_2 \ge 599$

$1 \le x_1 \le 6$

$1 \le x_2 \le 6$

The extreme points of the feasible region are

A_1 $(1, {}^{13}/_8)$, B $({}^{17}/_{12}, {}^{21}/_{16})$ and C_1 $({}^{311}/_{156}, 1)$

The values of the objective function are

$f(1, {}^{13}/_8) = -{}^3/_2$, $f({}^{17}/_{12}, {}^{21}/_{16}) = -{}^{13}/_{12}$, $f({}^{311}/_{156}, 1) = -{}^{77}/_{78}$

∴ The optimum solution is (311/156, 1)

We take $$x_3 = \begin{bmatrix} 311/156 \\ 1 \end{bmatrix} = \begin{bmatrix} 1{\cdot}994 \\ 1 \end{bmatrix}$$

Now $g(x_3) = g(1{\cdot}994, 1) = (1{\cdot}994 - 4)^2 + 2(1 - 3)^2 - 12 = 0{\cdot}027 < 0{\cdot}03 \in$

Hence, the optimum solution is given by $x_1 = 1{\cdot}994$, $x_2 = 1$

9.5 Summary

Among all the methods of constrained optimization here we have considered only the cutting plane method. The method is explained with the help of an example.

9.6 Self Assessment Questions

Using cutting plane method

Maximize $f = 7 - 2x_1 - 4x_2$

subject to $(x_1 - 4)^2 + 2(x_2 - 3)^2 - 12 \geq 0$

$x_1 + 2x_2 - 6 \leq 0$

$1 \leq x_1, x_2 \leq 6$

with the tolerance as $\in = 0{\cdot}3$

Using cutting plane method

Maximize $f = 1 - 4x_1 - 2x_2$

subject to $2(x_1 - 2)^2 + (x_2 - 3)^2 - 12 \geq 0$

$2x_1 + x_2 - 3 \leq 0$

$0 \leq x_1, x_2 \leq 5$

with $\in = 0{\cdot}2$

References

1. Linear Programming and Game Theory : Chakravorty & Ghosh ; Moulik Library

2. Operations Research : Kanti Swarup, P. K. Gupta, Man Mohan ; Sultan Chand & Sons

3. Operations Research : J. K. Sharma ; Mackillan India Limited

4. Linear Programming and Theory of Games : S. D. Sharma and Hemlata Sharma ; Keder Nath Ram Nath & Co

5. Operations Research : R. K. Gupta ; Krishna Prakashan Mandir

6. Linear Programming and Theory of Games : P. K. Gupta ; Khanna Publishers

7. Optimization theory and Applications : S. S. Ro ; Wiley Eastern Limited

8. Optimization method in operations research : K.V. Mital & C. Mohan ; New Age International Publishers

9. Operations Research (In Introduction) : Hamdy A Taha Prentice Hall of India Private Limited.

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—রবীন্দ্রনাথ ঠাকুর

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Price : Rs. 225.00

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Published by Netaji Subhas Open University, DD-26, Sector-I, Salt Lake, Kolkata - 700064 & Printed at Gita Printers, 51A, Jhamapukur Lane, Kolkata-700 009.